ARITHMETIC

IN

THE ANCIENT ORDER,

FULLY, YET FAMILIARLY, DEMONSTRATED;

FACILITATED IN THE STUDY BY AN ADAPTATION TO RECITATION IN CLASSES;

SIMPLIFIED IN PRACTICE BY CONCISE MAXIMS AND MODES OF STATEMENT IN PROPORTION.

PREPARED FOR SUPERIOR SCHOOLS;

AND FORMING

A COMPLETE GUIDE TO SELF-INSTRUCTION
BY PERSONS BEYOND THE TERM OF PUPILAGE.

By JOSEPH P. BARTRUM.

BOSTON:
WILLIAM D. TICKNOR.
1837.

Lately published by the same author, in a neat pocket volume,

THE PSALMS, NEWLY PARAPHRASED FOR THE SERVICE OF THE SANCTUARY. Designed chiefly as a Supplement to Sacred Lyrics in Use.

CAMBRIDGE:
FOLSOM, WELLS, AND THURSTON,
PRINTERS TO THE UNIVERSITY

TO THE

HON. JOHN PICKERING,

THE SOUND LOGICIAN, THE LEARNED PHILOLOGIST,

THE AMIABLE MAN,

THIS EFFORT LOGICALLY TO DEMONSTRATE THE PRINCIPLES

OF A FAMILIAR, BUT IMPORTANT, SCIENCE,

AND

TO FACILITATE THE APPLICATION OF IT TO PRACTICE,

IS

MOST RESPECTFULLY

DEDICATED

BY THE

AUTHOR.

PREFACE.

It may well create an expression of surprise, that any one should now bestow his time, and risk his property, in preparing and publishing a new book on arithmetic; the schools run over with *arithmetics*, and the booksellers' shelves groan under their weight. But are they really new? One undoubtedly is; a work *sui generis;* and if mine also can make pretensions to novelty, though formed on the ancient plan, the great outline of which I deem essential to a just application of the science, it may perhaps gain the voice of a few who are unwilling to believe, that every thing which is old is necessarily evil.

Buried within the forests of the West, it was the author's lot to while away not a few weary months, as they otherwise at least must have been, by engaging in the task of tuition in a private family. Books were few; those on arithmetic, in the writer's judgment, miserably defective; and when he as little expected to be writing on the arithmetic of the Europeans, as on that of the Chinese, he found himself engaged in composing rules, and forming demonstrations, that were at length to make a book. What degree of originality that book may claim on a subject so battered and bare-worn, every one who looks into it will determine for himself; certainly when its author formed the rude outline, it was without a model, and none has he since taken; the few treatises he has since examined have not commended their plans to his judgment, although they have led him deeply to consider his own, and occasioned his entire abandonment of the old proportional *form*. The

author's method of teaching differed from any thing he had then heard of in arithmetic; to himself it was original; and its application to his pupils proved eminently successful, to the extent to which it was carried. Not that the interrogative mode of instruction can at this day be original to any man in the sense of a first invention; it is as old as Socrates; and, to the author's knowledge, was scholastically exemplified forty years ago in a printed volume of questions to a famous Latin grammar. But such is the arrogance and folly of some celebrated foreign names, that they build their pretensions to fame on their own ignorance of what has been done before them. The author's method was, recitation on interrogatory from first to last, strict demonstration of every step of a process, daily rehearsal of the rule under exemplification, and a perfect knowledge of one rule before another was entered upon.

The excessive labor of separately examining the work of every scholar has been the chief cause of the long prevailing inefficient mode of teaching arithmetic on the ancient plan. To remove this annoyance entirely, the author proposes that scholars in arithmetic, as in other departments of learning, shall be formed into classes, that they shall be assigned a common task, and recite together. To give this method its greatest efficiency, he has thought the interrogative form eminently the best in a science which has no isolated facts, but wherein the dependency and succession of all the parts must be known, or nothing is known. One serious objection may be urged to most interrogative systems of instruction; namely, that, the discussion being divided between teacher and learner, the consequence is, that the answers of the learner are so many broken sentences, conveying little or no meaning as they stand alone; therefore, never recollected. No similar objection can exist to the interrogatory scheme here given; the learner discusses

and explains every thing; the questions are so formed, as to enunciate only the points to be considered; and every answer is a complete proposition, though not always a complete syllogism; since, for the ease of the pupil himself, the premises and the conclusion of an argument are often separated into distinct answers. It may be thought, that to commit so much to memory as this book contains must prove a very serious task. The author confesses that he has not undertaken to write a toy-book, but one that shall afford instruction to a learner as long as he shall live. We know not what memory can do till we exercise it; we throw away our faculties from the non-exercise of them; and the fatigue, if it be such, of learning by rote is greatly mitigated by a participation with others, whose recollection may supply our defects, while our own supplies theirs. On persons not pupils this labor will, of course, not be self-imposed; nor is it necessary that answers should be given by pupils in the very words of a book; if the learner can give an explicit answer of his own, it may be better than what he has failed to recollect. This observation however does not apply to the definitions, maxims, and conciser rules, here given: these ought to be repeated *verbatim*. In all of them, utility rather than logical nicety, though never to the neglect of logic, has been the writer's aim. Laboring to be brief, he trusts that he is not obscure; endeavouring to be choice in his expressions, he hopes they are, at the same time, comprehensive. He has been peculiarly anxious to omit no case that can possibly occur in the performance of the fundamental operations, fractional or integral; that the student may never be at a loss where to look for information.

To return to the manner of recitation, let the instructer carefully avoid observing any particular series in the commencement of exercises; let it be now senior in a class,

now junior, and so on; that all, knowing their liabilities, may study all the lesson appointed to them. In operations, let the instructer have before him the slate of one known to be habitually correct, and let every figure, uniformly, be recited, by the successive apportionment of parts of an operation and proof to different members of the class, and an error be called out upon by any or all who notice it. Whenever there is a just suspicion of duty neglected, a slight inspection of the slates will suffice for detection; and to conclude the subject of recitation, let the italicized propositions, the definitions, maxims, and shorter rules, be daily repeated, in connexion with the examples wrought under them.

That a demonstrative science should ever have been taught as an affair of memory only, is not so much to be reckoned among the vagaries of the human mind, as to be attributed to an innate love of ease. Yet if mathematical studies bring any advantage with them over other literary pursuits, it consists in habituating the mind to strict investigation; and it is melancholy, that so little use should have been made of the every-day process of teaching arithmetic, to form this important habit; a habit infinitely more valuable than ability to sum, cipher, and cast up accounts. One most precious opportunity which God has given to man for the enlargement of his faculties is thus perpetually thrown away; a manly taste and vigorous judgment are seldom formed; and the consequence is, that a large portion of our earlier years is spent amid the dreams of fiction, often blasted by the poisonous influence of falsehood.

Another mischief arising from the neglect of demonstration, is the widely spread belief, that mathematical differs from moral evidence in its very nature; so much so, that the latter can lend, it is imagined, but a feeble and dubious light to the inquirer, while all is blaze and certainty to the

mathematician. Whoever will accompany the author through the processes of this book, may find his opinions strangely altered on that subject: he will find that all evidence is reducible to the same principles of sound logic; and that without a continual resort to grounds of argument common to every pursuit, no system of accounts could ever have been formed, none can now be explained. Artificial demonstration may answer the purpose of the mathematician, when his foundations have been laid; the foundations themselves must rest on an appeal to the common sense of man, or the fabric reared will prove but an airy vision. Mathematics have more self-evident truths than morals, because their elements are unclouded by adventitious circumstances; we see them stripped and bare; but, like the timbers of a building, they can be put together only by bringing into action the same powers and processes of reasoning which conduct us to just, and, in ten thousand cases, equally certain, conclusions in every other department of human knowledge. This is strikingly the case with *proportion;* an idea that mixes itself up, from our earliest years, with almost every transaction of human life; by which we judge of quality as well as of quantity; the exclusion of which, were it possible, would bring all human affairs into endless confusion. Proportion is made the student's guiding star through the following work; for it must be reasoned upon in arithmetic long before we come to any strict definition of it. The author thinks that he has given explanations, and hit upon a device, suggested by the algebraic notation, that will indelibly imprint its most important maxims on the mind; and he is desirous that the student, whenever at a loss, should mentally sum up the doctrine and its application in answers to the following queries:

What is proportion?
Between what kind of things does it exist?

What are its uses?

What is the manner of its use?

How is the justice and correctness of the proportional operation shown?

What are the maxims that guide us in its application?

The student who can answer these inquiries will not think of dignifying the laborious efforts of the uninstructed, to pick out the value of a single article in small transactions, with the philosophical name of analysis.

The author is indebted for all he knows of arithmetic to Bonnycastle, a treatise comparable to whose he has never seen, and he has not seen this for some thirty years. His chief aids have been Webber, of Harvard College, and Hutton, the celebrated professor of Woolwich: deficiencies of explanation in their works he has endeavoured to supply; they suppose a knowledge far beyond what the youthful, what, in many instances, the advanced, student can possess; and where a single link is wanting in the chain of demonstration, the mind feels itself insecure. The definitions with which we begin the work are beyond most beginners in arithmetic; but if remembered, they may hereafter be understood.

The remaining topic of consideration in the mode of studying arithmetic is scarcely less important than those already noticed. The student must not advance a step beyond the ground which he has made his own; he is subject indeed to lose his acquisitions by the failure of recollection, and this is the principal reason why he should ever retrace his steps. The advantage of thus proceeding, *and thus delaying*, will be found immense by every instructer, whose patience and whose sense of duty will suffer him to pursue the plan. How poor the satisfaction, how futile the results, of persuading parents, that their children have made a progress which a slight examination will disprove!

It cannot indeed be expected, that the perfect demonstration herein attempted of every arithmetical process will be fully understood on a single perusal; but the author flatters himself, that if a youth begins upon this work with ability to reckon the ordinary varieties of change in a dollar (and this he requires, but requires no more); if he gives a fair attention to it, under a faithful instructer, he will, at the conclusion of a second complete study of the work be entitled to the name, because possessing the skill, of an accountant; to be perfected, of course, by the knowledge and practice of book-keeping, if his pursuits lead that way.

For accountants, in the largest sense of that term, the author writes; he writes not for mathematicians; he has not ability to soar with them to the celestial heights, and to measure the earth and the ocean; all he pretends to is, some share of common sense, and a disposition to investigate, to its remotest boundaries, whatever subject necessity or curiosity may lead him to employ his thoughts upon. If he sees a defect, it is his humor to aim at supplying it; if an absurdity, to essay its removal. Let this generally account for variations from accustomed terminology and modes of explanation to be found in the following work. It may be ignorance that urges him, it may be rashness and presumption that impel him, to make innovations on established names and forms; but he cannot help it; it is his fate to do so, and his fate perhaps to be condemned. What, after all, can a man strive to do, that is worth striving for, but to do good, should he even be mistaken in his means for effecting it? always supposing him to be honest and true-hearted. Without a particle of faith in the efficacy of subversion, but to do mischief, a few cases of morals only excepted, he believes that one generation may stand on the shoulders of another, and see a little way beyond, and make a little advance in the path of eternal improvement.

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But not only will the manner of this treatise furnish ground of reprehension, the matters also introduced into it may offer an abundant theme of condemnation. The author's reply in anticipation is, that he lives and writes in no ordinary times. Shall not a man who writes for youth in an age of infidelity take every fair opportunity which his subject offers, to imbue their minds with sentiments of piety and virtue? Shall he not in our day fortify them with principles of defence against the foul spirit of atheism that is going about, through the civilized world, seeking whom it may devour; carrying wretchedness, and sin, and death, into every abode that gives it shelter? Vain is the education of the intellect, if the heart be neglected! In season therefore or out of season, the author thinks he acts not unseasonably, in bringing plain, manifest truths of natural and revealed religion, springing fairly from his subject, to the consideration of the youthful mind; that it may not be misled into the mazes of skepticism on subjects easily understood; that it may realize the presence of a spiritual as well as of a material world; and that the flimsy sophistry, too often admitted as argument by men whose excess of candor leads them to make every possible concession to a ruthless foe, may have no weight with it in deciding on truths essential to the existence of any religion in the world.

The author has been free in the use of words that form the terminology of the science, and has perhaps been guilty of adding some thereto. Either peculiar terms must be employed, or circumlocution must be endless; though indeed there is a third horn to this dilemma, on which we shall be still more likely to impinge; it is that of learning nothing. It is a perfect delusion, the mere cant of a sciolist, to talk of teaching a science without the use of peculiar terms. Learning consists very much in a power of generalization; and if we cannot generalize our expressions, we shall never

be able to generalize facts. A single term fitly employed will often convey the meaning of a long period; in order to this fitness, however, there must be uniformity of meaning; and in this matter the author considers arithmetic, as it now appears, to be very faulty. Let us take a few examples. The word *compound* is applied to multiple numbers, and to numbers of denomination. *Rate*, beside including estimates of every possible kind, is the estimate both of a hundred and of unity, in the very same kind. *Discount* is an allowance made on all sorts of principles. The word first mentioned will not be found in the terminology of this treatise, except as distinctive of interest and discount, of particular kinds; rate is always, in unmixed money matters, the estimate of a hundred, when not expressed to be the rate of unity; one important kind of discount is here discriminated by the addition of the word *trade*; and the most important, by the word *proper*. Some names too without meaning, or with a meaning altogether inappropriate, are exchanged for others that express what they do mean; as the strongest instance, the author mentions the name *alligation*. It was a clumsy contrivance at any time, to tie numbers together, lest they should run out of mind; but if the contrivance be dismissed, there can exist no reason for retaining the name, except that of its having been long used; and if inelegancies and absurdities are to go down to all posterity, it will continue to be used.

The author professes to write for superior schools, because he cannot conceive of any method of teaching accounts, that shall not include fractions as a fundamental part of arithmetic, and a perfect knowledge of them as indispensable to the practice of it; and how this knowledge is to be communicated in common schools, is a question equally hard for him to determine. The abandonment alto-

gether of written arithmetic in some of these schools, he thinks a most pernicious innovation; while recitation and operation go hand in hand, something will be learned, even should it be little.

In the conducting of proportional operations, the author has confined himself strictly to the exhibition of a ratio and an estimate. He believes the double rule, and the varied statement, now in one form, then in another, of the ancient systems, to be the great source of perplexity attending the rule of three; the directions, too, for distinguishing the terms are commonly so loose as to afford little aid to the learner who is not gifted with uncommon powers of discrimination; and frequent failures at school precede a total abandonment of any rule of proportion in after life. The calculations indeed essentially incident to business are gotten through; and men often imagine they have made rules for themselves, irreducible to system, yet certain in result. If any one is willing to content himself with such knowledge as this, when it is in his power to obtain clear ideas, and to calculate on fixed and intelligible principles, he is welcome to remain in his ignorance, and to risk the consequences of his folly. Every operation governed by the principle of proportion is, in the present work, reduced to the proportional form. When the author began on this department, he had not the remotest idea that he should be able to follow up the windings of some of the ordinary rules to their spring-head, the fountain of proportion. Whether these processes are so exhibited elsewhere, he cannot pretend to say; he has never seen it done, and to do it himself has cost him infinite labor: he has done it truly, he believes, and demonstratively; from the first to the last page of proportional arithmetic, ever perhaps opened in the counting-room; yet he must not quit this subject, without expressing his

fears, that some of his formulas and maxims are conveyed in language not of perfect mathematical propriety.

To the candid judgment of the mathematician, if any such shall deign to look into it, this work of an author unskilled in mathematical lore is submitted; to the aspirant after knowledge in a humbler sphere, it may prove no ungrateful offering. It is committed to a press, the name of which inspires confidence; the author's unaided efforts in the composition, and his unavoidable absence from the press, will be taken in excuse, he trusts, for unavoidable imperfections.

CONTENTS.

SYMBOLS.

To be learned after Subtraction.

What are symbols? — Symbols are signs, or concise modes of representation to the eye.

What signs do we employ in arithmetic? — In arithmetic we employ signs of number, of operation, and of proportion.

What are signs of number? — The signs of number are figures.

What are the two other kinds? — The signs of operation and proportion are marks, by which we denote a proportion and distinguish an operation.

What is the sign of addition? — The sign of addition is a perpendicular cross; to read it, we say *plus*, or *and*. — $+$ *Addition.*

What is the meaning of those words? — *Plus* is a Latin word, signifying *more*; *and* is from the verb *to add*.

What is the sign of subtraction? — The sign of subtraction is a short horizontal line; to read it, we say, *minus*, or *less*; *minus* signifying *less*. — $-$ *Subtraction.*

What is the sign of multiplication? — The sign of multiplication is an oblique cross; to read it, we say *into*, or *multiplied into*. — $\times$ *Multiplication.*

What is the sign of division? — When the divisor precedes the dividend, the sign of division is a colon; it is read *in*, or *dividing*; when the divisor follows the dividend, it is a short horizontal line passing between the dots of a colon, and is read, *divided by*; division is also represented fractionally, by a short horizontal line between two numbers, above and below. — $:$ *Divisor before Dividend.* $\div$ *Divisor after Dividend.* $\frac{n}{d}$ *Division fractionally represented.*

What are the signs of proportion? — Proportion of any two numbers has the same signs with division, and is read, *as the divisor to the dividend, as the denominator to the numerator.* — $:$ $\frac{n}{d}$ *Proportion.*

What is the sign of equality? — The sign of equal numbers is two short, equal and parallel, horizontal lines; it is read, *equal to.* — $=$ *Equality.*

What is the sign of roots? — The sign of arithmetical roots is two short, unequal lines, meeting below at an angle. — $\surd$ *Roots.*

What other signs are there in arithmetic? — Other signs relate to the numbers called compósites.

" Wolfius makes the sign of division two dots," but places the dividend on the left; a most inconvenient arrangement in practice, from the necessity of figuring the quotient toward a limit, or of placing it under the divisor, which latter mode destroys the form of the equation.

ARITHMETIC.

PART I. INTEGERS.

NOTATION. NUMERATION.

General Definitions.

What is arithmetic ? — Arithmetic is the science of numbers.

What is a science ? — A science is an orderly system of instruction concerning things purely or chiefly intellectual.

What is number ? — Number is an expression of individuality, or separate existence.

What are numbers ? — Numbers are certain words and figures by means of which we reckon individual or separate things.

What are the words ? — The words are, one, two, three, four, five, six, seven, eight, nine, nought, &c.

Can nought be a number ? — Nought is the privation of number ; it is used however as the name of a figure, which aids us in numbering.

When are figures employed ? — In accounts, for compactness' sake, we use figures, named from the words just mentioned.

How many figures are there, and how written ? — We use as many figures as we have fingers ; they are written thus : 1 2 3 4 5 6 7 8 9 0.

By whom were these figures invented ? — Some of our present figures have been traced to the ancient Egyptians ; they were extended probably, and improved, by the Arabians, or Saracens, who possessed themselves of Egypt under the Califs ; hence the name of Arabic figures.

See Russell's Egypt, in the Family Library.

These figures are sometimes called digits ; what is the meaning of that term ? — *Digit* is a Latin word abbreviated, signifying a finger, pointer, indicator.

Whence may the application of this word have been derived? — The ten fingers, it can hardly be doubted, have every where first suggested a mode of reckoning.

Are any other characters used to indicate number? — The ancient Romans used characters resembling their letters: these are called Roman numerals.

Are figures of different kinds? — Figures are of two kinds, significant and ciphers.

What is the first mentioned kind? — Significants signify, or represent, number.

What is the second kind? — Ciphers represent the absence of number from a particular place.

Cifra, in later Latin writers, signifies any note of number; hence our verb, to cipher.

By what other name is the absence of number denoted? — *Zero* is an Italian word also used to signify nought.

What is meant by place? — Arithmetical place is the position of a figure in relation to any other, and to all others, appertaining to the same number.

When are ciphers employed? — Places necessary to be noted, and not filled with significants, are marked with a cipher.

Is there any distinction of numbers? — Numbers are distinguished, by the value of the things they represent, into two kinds, called integers and fractions.

What is the meaning of integer? — *Integer* is a Latin word, signifying whole, entire.

What is a whole number? — A whole number represents a thing entire; as a dollar.

By what other name are entire things known? — A unit also signifies one entire thing.

How does integer differ from unit? — A unit designates only one thing; an integer may denote several things represented by one figure, as 9.

What is a fraction? — A fraction represents part of one entire thing; as half a dollar.

What is the meaning of the term? — Fraction is from a Latin word, and signifies a piece broken off, or its numerical representation.

How are fractions figured? — Fractions are commonly denoted by figures above and below a short line; as one half ($\frac{1}{2}$).

Are there any other distinctions of number? — There are simple numbers and composites.

What are the first mentioned ? — Simple numbers represent things having names not interchangeable ; as so many horses.

What are the second ? — Composites, usually called compound numbers, represent things capable of receiving different denominations ; as one dollar, one hundred cents.

Can you read figures, from one to one hundred? — 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 92, 93, 94, 95, 96, 97, 98, 99, 100.

Can you recite those numbers by heart ? — [Recites.]

The learner should be able to recite numbers orderly, and to read figures promiscuously, from 1 to 100, before proceeding farther.

Definition.

What is the first arithmetical operation ? — The first operation in arithmetic is called numeration.

What is it to enumerate ? — To enumerate is to reckon, to number.

How do we make others acquainted with the number of any thing ? — We communicate number as we do most other things, by speech or by writing.

How many figures are there ? — There are ten figures.

How many numbers are there ? — Numbers are, to us, without end.

How then can we make numbers of any amount to be known by means of ten figures only ? — By arranging figures in different places we vary the expression of their value, and can thus express any number whatever.

How are the places of figures distinguished ? — Different places have different names.

But how is the value of a figure known, when its place is not named ? — The place of a figure is seen and known by the order in which that figure stands.

How is the order itself ascertained ? — The order of places is fixed by rule.

What is a rule ? — A rule is a precept for governing our conduct.

What is a rule in arithmetic ? — An arithmetical rule is a prescribed method of writing and reciting numbers ; of conducting also and reciting numerical operations.

What then must guide you in arithmetic ? — The rule must ever be our guide.

Are you now able to define numeration? — *Numeration is the writing and reciting of numbers in an orderly arrangement of places, fixed and named by rule.*

Place.

What is meant by a rank of figures? — A rank of figures consists of any three places, beginning from the right of whole numbers, and often marked by a separating comma.

Can you give me an example? — If we set down the figures, 6 5 4, 3 2 1, a comma between 4 and 3 separates two ranks.

Is this separating mark necessary? — A comma to distinguish rank is unnecessary, and tends rather to confusion than distinction; separators being necessarily used for other purposes.

What is the first rank of figures called? — The first rank of figures is the rank of units.

But if a unit be one thing, how can we speak of units? — All things are, separately, each a unit; collectively, they are units, in the plural number.

Where is the place of units? — In a line of figures the place of units is on the farthest right of whole numbers.

What is the second place of units called? — The second place of units is called tens; because the significance of a figure is increased tenfold for every place it is removed from the right toward the left.

Can you give an example? — The figure 2, when on the farthest right, signifies 2 units; advanced to the second place toward the left, it signifies 20 units; to the third place, 200 units.

What name is given to this last place? — The third place of units is called that of hundreds.

How is the second rank named? — The second rank of figures is called the rank of thousands.

What are thousands made up of? — Thousands, and all other numbers, are made up of units; for a thousand signifies a thousand units.

What are the second and third places of the second rank called? — The second and third places of thousands are called tens of thousands, and hundreds of thousands.

What is the third rank? — The third rank is that of millions.

How are its three places named? — Its places are, millions, tens of millions, hundreds of millions.

What are the fourth and fifth ranks? — The fourth and fifth ranks are those of billions and trillions.

The French mode of counting the series from millions upward is so concise, and so consistent with the previous order, that I hesitate not to adopt it.

In what place of figures is the first place of billions? — The first place of billions is the tenth place of figures.

What numbers are expressed by the names, eleven, twelve, to nineteen? — The name eleven comprehends ten and one; twelve comprehends ten and two; thirteen are the words three and ten abbreviated and compounded; fourteen are four and ten; fifteen are five and ten; sixteen are six and ten; seventeen are seven and ten; eighteen are eight and ten; nineteen are nine and ten.

What are numbers ending in *ty* expressive of? — *Ty* seems to be an abbreviation of ten; since twenty is used for two tens: thirty, for three tens; forty, for four tens; fifty, for five tens; sixty, for six tens; seventy, for seven tens; eighty, for eight tens; ninety, for nine tens.

In a line of numbers, what places must those expressive of tens occupy? — All figures representative of tens, and no more, occupy the two farthest places to the right, in a line of whole numbers.

Why do you say of whole numbers? — Because fractions are placed still farther to the right.

In figuring, on which hand do you begin? — In writing or reading a line of figures, we usually begin on the left.

When enumerating where do you begin? — The places of figures are commonly enumerated from the right; because the farthest place of whole numbers is easily recollected to be that of units.

Can you now recite the table of numbers?

NUMERATION TABLE.

2d, or rank of thousands.			1st, or rank of units.		
Hunds. of T.,	Tens of T.,	Thousands,	Hundreds,	Tens,	Units.
6th place,	*5th place,*	*4th place,*	*3d place,*	*2d place,*	*1st place.*

	4th, or rank of billions.			3d, or rank of millions.		
Trillions.	Hunds. of B.,	Tens of B.,	Billions,	Hunds. of M.,	Tens of M.,	Millions,
13th place,	*12th place,*	*11th place,*	*10th place,*	*9th place,*	*8th place,*	*7th place,*

Having recited the table from the right, can you also recite it, so far as it goes, from the left? — [Recites.]

Do you remember the place of millions? — Millions are in the 7th place.

Let the learner be questioned in the places promiscuously, till he is familiar with them.

In enumerating a rank, whence do we begin? — A rank is commonly enumerated from the place which gives the rank its name.

If I say tens, hundreds, thousands, do I enumerate a rank? — Tens, hundreds, and thousands, are parts of two ranks.

When I say hundreds of thousands, tens of thousands, thousands, do I enumerate a rank? — To say hundreds of thousands, tens of thousands, thousands, is to enumerate a rank from its highest place.

What is the name of the 2d rank? of the 5th? of the 1st? of the 4th? of the 3d?

Demonstration of rule.

On which hand do we begin to write? — We begin to write on the left, unless some special purpose require a contrary mode; as sometimes is the case in figures.

Where is the highest place of figures? — The highest place in a line of figures is on the left.

Why so considered? — The left is made the highest place of figures, because they are lowered in estimate, to a tenth part, for every succeeding place to the right.

Can you show this by example? — If 1 be in the third place toward the left, it stands for 100; removed to the second place, it stands for 10 only, a tenth part of 100.

In writing numbers from the left, what other order do we observe? — In writing and reading numbers, we observe the order of the numeration table.

Why? — Because the numeration table assigns to every particular place of figures a particular value.

Must every place noted in the table be set down? — From the highest number mentioned, as to be written, every place must be set down which is noted in the numeration table.

Why? — Unless every place from the highest be set down, the true place of a figure cannot be known; there being no way of learning this but by reckoning from, or to, units.

Can we constantly fill places with significants? — Places cannot always be filled with significants; for things are of every number, and so computed; hundreds without the mention of tens, tens without any mention of units.

What is then done? — Every place below the highest, and not mentioned, must be filled with a cipher.

Can you now, in a few words, give me the rule?

RULE OF NUMERATION.

To enumerate in writing, begin on the left, with the significant of the highest place mentioned; set down the significants of succeeding places in the order of the numeration table, marking every place omitted to be mentioned in that order with a cipher.

Consideration of proof.

How would you prove the correctness of an enumeration? —I prove that numbers are rightly figured, by showing their agreement with the numeration table.

Can you exemplify this? —If the sum mentioned be hundreds, I show that the figure representing it is in the place of hundreds; and so of any number.

How do you know the numeration table itself to be right? — Any table of numbers must be right, which enables us to count truly.

Does our table always enable us to do this? — We find, that when we enumerate according to the table, and have not set down a wrong figure, no mistake is ever made.

What is meant by setting down a wrong figure? — Errors are sometimes committed in accounts, and a number set down greater or smaller than the true one; as a 9 for a 7.

APPLICATION.

Suppose, scholar, you are directed to write in figures the number thirty-five. Thirty are three tens, and tens are the highest place mentioned; therefore, according to the rule, 3 are to be set down on the left, for the highest place, or tens, and 5 to be annexed thereto, for the units; thus we have 35. Suppose the number mentioned to be four hundred; 4 are set down on the left for the hundreds, with two ciphers annexed for the tens and units not mentioned; thus we have 400. Suppose it to be six thousand nine hundred and seven. The highest place requires a 6; the hundreds, a 9; the tens, a cipher, for no tens are mentioned; the units require a 7; all making together, 6907. Let it be a million eighty thousand four hundred and one. The million is figured by 1; the hundreds of thousands by 0, for none are mentioned; the tens of thousands by 8; the thousands by 0; the hundreds by 4; the tens by 0; the unit by 1; making 1080401. Compare these figures with the table and the rule, for your own satisfaction; let the rule be your guide, and whenever you are at a loss, recite it mentally.

In figuring the examples that follow, you are not to affix names on the slate, but to recite the figures, and their amount, after this manner. Suppose the example to be the last we illustrated; then say, one, nought, eight, nought, four, nought, one; in all, seven places; one million eighty thousand four hundred and one.

This mode of recitation makes it unnecessary to give examples in figures, to be set down in words; for thus the figuring and naming of numbers are learned at the same time.

Numbers to be figured on the slate, and recited.

1. Twelve. Twenty-eight. Fifty-six. Thirty-nine. Seventy-two. Ninety-three. Seventeen. Twenty-one. Thirty.

2. Fifty. Sixty-eight. Ten. Forty. Forty-three. Eighty. Eighty-eight. Nineteen. Seventy-five. Ninety-two.

3. One hundred sixteen. Three hundred seventy-eight. Seven hundred eleven. Four hundred twenty-nine.

4. Five hundred six. Two hundred one. One hundred two. Seven hundred forty. One hundred eleven.

5. Nine hundred seventeen. Three hundred eight. Four hundred. Two hundred ten. Five hundred sixty-three.

6. One hundred one. One hundred ten. Seven hundred fourteen. Five hundred fifty-five. Nine hundred nine.

7. Three thousand nine hundred four. Seven thousand twenty-seven. Four thousand three. Six thousand eleven.

8. Five thousand five. One thousand eight hundred thirty-two. Six thousand five hundred. Nine thousand one.

9. Ten thousand three hundred eighty-seven. Fifty thousand one hundred ninety-three. Forty thousand five.

10. Twenty-eight thousand forty. Seventy thousand four. Nineteen thousand sixty-eight. Thirty thousand eleven.

11. Four hundred thousand seven hundred ninety eight. Seven hundred sixty-three thousand forty-five.

12. Nine hundred seventy thousand. Two hundred ten thousand thirty-six. One hundred thousand one.

13. Six million. Three million fourteen thousand five hundred twelve. Seven million ninety-five. Ten million ten.

14. Eighteen million six hundred thousand five hundred ten. Twenty million four hundred five thousand fifty.

15. Two hundred forty-nine million seven hundred one thousand eight hundred fifty-three.

16. Six hundred million twenty-three thousand six. One hundred million one hundred one. A thousand million.

17. Five billion nine hundred million two hundred sixty-four thousand five hundred six. Fifty thousand million.

18. Sixty billion thirty-three million one hundred eight thousand seven hundred. Five hundred thousand million.

19. Nine hundred two billion forty-eight thousand seven hundred twenty-two. A thousand billion.

20. One trillion twenty-four billion six hundred seven million seven thousand seven.

These or other numbers should be repeatedly figured by the learner, till his figuring and recitation are perfect. The number of days or weeks this may consume is not of the smallest importance. The examples of millions exceeding their proper rank are introduced, that the learner's attention may be called to the different manners of expressing like values.

ADDITION.

Definition.

What is the second arithmetical operation? — **The second operation in arithmetic is addition.**

How are things increased? — **Things are increased by adding to them.**

What is the act, in making addition? — **In making addition we put things together which before were separate.**

Can you give me an example? — **Separate heaps of dollars may be put together: this is addition.**

Is every heap increased by this means? — **Not every heap, but one only, is thus increased; for it then comprehends all.**

What is the meaning of the word comprehend? — **To comprehend is to take together; to consider as united what before was distinct.**

What is the meaning of the word sum? — ***Sum* signifies amount, value.**

How is the word applied in addition? — **The amount of several heaps of dollars put together would be called their sum, or sum total.**

Can you now define addition? — ***Addition is the increase of a number, by comprehending with it numbers before separate, in a sum, called the sum total.***

Abstract and concrete numbers.

What do numbers represent? — **Numbers always represent things, their measures, or turns.**

What do you mean by turns? — **By turns I mean repetitions of the same things, or the several times that things of the same kind and number are taken; as so many dollars, once or oftener.**

What is meant by abstraction? — **Abstraction is a metaphysical term, signifying the consideration of a quality apart from its substance; it may also be used to denote the consideration of a sign apart from the thing signified.**

How is the mind engaged when occupied about number? — **When occupied about number, the mind thinks of figures, their names, or the things they represent.**

Are figures a quality of any thing? — Arithmetical figures are not a quality but a sign.

What then may figured numbers be called? — Figured numbers without particular designation, and considered as signs apart from any thing capable of being signified by them, may be called abstract numbers.

What is the meaning of concrete? — Concrete signifies the quality of growing together, of uniting.

What then may numbers denoting realities be called? — Numbers representative of real objects may be called concrete numbers; because the sign and the thing signified are united in our thoughts.

This subject is undoubtedly beyond the stage we are at; but it may be returned to hereafter, when it will be better understood.

Can you recite the table of addition?

ADDITION TABLE.

To be recited orderly till the scholar is perfect, then promiscuously on examination.

1	and	nought is	1	2	and	6 are	8	3	and	12 are	15
1	and	1 are	2	2	and	7	9	3	and	13	16
1	and	2	3	2	and	8	10	3	and	14	17
1	and	3	4	2	and	9	11	3	and	15	18
1	and	4	5	2	and	10	12	3	and	16	19
1	and	5	6	2	and	11	13	3	and	17	20
1	and	6	7	2	and	12	14	3	and	18	21
1	and	7	8	2	and	13	15	3	and	19	22
1	and	8	9	2	and	14	16	3	and	20	23
1	and	9	10	2	and	15	17				
1	and	10	11	2	and	16	18	4	and	nought are	4
1	and	11	12	2	and	17	19	4	and	1	5
1	and	12	13	2	and	18	20	4	and	2	6
1	and	13	14	2	and	19	21	4	and	3	7
1	and	14	15	2	and	20	22	4	and	4	8
1	and	15	16					4	and	5	9
1	and	16	17	3	and	nought are	3	4	and	6	10
1	and	17	18	3	and	1	4	4	and	7	11
1	and	18	19	3	and	2	5	4	and	8	12
1	and	19	20	3	and	3	6	4	and	9	13
1	and	20	21	3	and	4	7	4	and	10	14
				3	and	5	8	4	and	11	15
2	and	nought are	2	3	and	6	9	4	and	12	16
2	and	1	3	3	and	7	10	4	and	13	17
2	and	2	4	3	and	8	11	4	and	14	18
2	and	3	5	3	and	9	12	4	and	15	19
2	and	4	6	3	and	10	13	4	and	16	20
2	and	5	7	3	and	11	14	4	and	17	21

4 and 18 are 22
4 and 19 23
4 and 20 24

5 and nought are 5
5 and 1 6
5 and 2 7
5 and 3 8
5 and 4 9
5 and 5 10
5 and 6 11
5 and 7 12
5 and 8 13
5 and 9 14
5 and 10 15
5 and 11 16
5 and 12 17
5 and 13 18
5 and 14 19
5 and 15 20
5 and 16 21
5 and 17 22
5 and 18 23
5 and 19 24
5 and 20 25

6 and nought are 6
6 and 1 7
6 and 2 8
6 and 3 9
6 and 4 10
6 and 5 11
6 and 6 12
6 and 7 13
6 and 8 14
6 and 9 15
6 and 10 16
6 and 11 17
6 and 12 18
6 and 13 19
6 and 14 20
6 and 15 21
6 and 16 22
6 and 17 23
6 and 18 24
6 and 19 25
6 and 20 26

7 and nought are 7
7 and 1 8
7 and 2 are 9
7 and 3 10
7 and 4 11
7 and 5 12
7 and 6 13
7 and 7 14
7 and 8 15
7 and 9 16
7 and 10 17
7 and 11 18
7 and 12 19
7 and 13 20
7 and 14 21
7 and 15 22
7 and 16 23
7 and 17 24
7 and 18 25
7 and 19 26
7 and 20 27

8 and nought are 8
8 and 1 9
8 and 2 10
8 and 3 11
8 and 4 12
8 and 5 13
8 and 6 14
8 and 7 15
8 and 8 16
8 and 9 17
8 and 10 18
8 and 11 19
8 and 12 20
8 and 13 21
8 and 14 22
8 and 15 23
8 and 16 24
8 and 17 25
8 and 18 26
8 and 19 27
8 and 20 28

9 and nought are 9
9 and 1 10
9 and 2 11
9 and 3 12
9 and 4 13
9 and 5 14
9 and 6 15
9 and 7 16
9 and 8 are 17
9 and 9 18
9 and 10 19
9 and 11 20
9 and 12 21
9 and 13 22
9 and 14 23
9 and 15 24
9 and 16 25
9 and 17 26
9 and 18 27
9 and 19 28
9 and 20 29

10 and nought are 10
10 and 1 11
10 and 2 12
10 and 3 13
10 and 4 14
10 and 5 15
10 and 6 16
10 and 7 17
10 and 8 18
10 and 9 19
10 and 10 20
10 and 11 21
10 and 12 22
10 and 13 23
10 and 14 24
10 and 15 25
10 and 16 26
10 and 17 27
10 and 18 28
10 and 19 29
10 and 20 30

11 and nought are 11
11 and 1 12
11 and 2 13
11 and 3 14
11 and 4 15
11 and 5 16
11 and 6 17
11 and 7 18
11 and 8 19
11 and 9 20
11 and 10 21
11 and 11 22
11 and 12 23
11 and 13 24

11	and	14	are	25
11	and	15		26
11	and	16		27
11	and	17		28
11	and	18		29
11	and	19		30
11	and	20		31

12	and	nought	are	12
12	and	1		13
12	and	2		14
12	and	3		15
12	and	4		16
12	and	5		17
12	and	6		18
12	and	7		19
12	and	8		20
12	and	9		21
12	and	10		22
12	and	11		23
12	and	12		24
12	and	13		25
12	and	14		26
12	and	15		27
12	and	16		28
12	and	17		29
12	and	18		30
12	and	19		31
12	and	20		32

13	and	nought	are	13
13	and	1		14
13	and	2		15
13	and	3		16
13	and	4		17
13	and	5		18
13	and	6		19
13	and	7		20
13	and	8		21
13	and	9		22
13	and	10		23
13	and	11		24
13	and	12		25
13	and	13		26
13	and	14		27
13	and	15		28
13	and	16		29
13	and	17		30
13	and	18		31
13	and	19		32
13	and	20	are	33

14	and	nought	are	14
14	and	1		15
14	and	2		16
14	and	3		17
14	and	4		18
14	and	5		19
14	and	6		20
14	and	7		21
14	and	8		22
14	and	9		23
14	and	10		24
14	and	11		25
14	and	12		26
14	and	13		27
14	and	14		28
14	and	15		29
14	and	16		30
14	and	17		31
14	and	18		32
14	and	19		33
14	and	20		34

15	and	nought	are	15
15	and	1		16
15	and	2		17
15	and	3		18
15	and	4		19
15	and	5		20
15	and	6		21
15	and	7		22
15	and	8		23
15	and	9		24
15	and	10		25
15	and	11		26
15	and	12		27
15	and	13		28
15	and	14		29
15	and	15		30
15	and	16		31
15	and	17		32
15	and	18		33
15	and	19		34
15	and	20		35

16	and	nought	are	16
16	and	1		17
16	and	2		18
16	and	3		19
16	and	4	are	20
16	and	5		21
16	and	6		22
16	and	7		23
16	and	8		24
16	and	9		25
16	and	10		26
16	and	11		27
16	and	12		28
16	and	13		29
16	and	14		30
16	and	15		31
16	and	16		32
16	and	17		33
16	and	18		34
16	and	19		35
16	and	20		36

17	and	nought	are	17
17	and	1		18
17	and	2		19
17	and	3		20
17	and	4		21
17	and	5		22
17	and	6		23
17	and	7		24
17	and	8		25
17	and	9		26
17	and	10		27
17	and	11		28
17	and	12		29
17	and	13		30
17	and	14		31
17	and	15		32
17	and	16		33
17	and	17		34
17	and	18		35
17	and	19		36
17	and	20		37

18	and	nought	are	18
18	and	1		19
18	and	2		20
18	and	3		21
18	and	4		22
18	and	5		23
18	and	6		24
18	and	7		25
18	and	8		26
18	and	9		27

18	and	10	are	28	19	and	6	are	25	20	and	2	are	22
18	and	11		29	19	and	7		26	20	and	3		23
18	and	12		30	19	and	8		27	20	and	4		24
18	and	13		31	19	and	9		28	20	and	5		25
18	and	14		32	19	and	10		29	20	and	6		26
18	and	15		33	19	and	11		30	20	and	7		27
18	and	16		34	19	and	12		31	20	and	8		28
18	and	17		35	19	and	13		32	20	and	9		29
18	and	18		36	19	and	14		33	20	and	10		30
18	and	19		37	19	and	15		34	20	and	11		31
18	and	20		38	19	and	16		35	20	and	12		32
					19	and	17		36	20	and	13		33
					19	and	18		37	20	and	14		34
19	and	nought	are	19	19	and	19		38	20	and	15		35
19	and	1		20	19	and	20		39	20	and	16		36
19	and	2		21						20	and	17		37
19	and	3		22						20	and	18		38
19	and	4		23	20	and	nought	are	20	20	and	19		39
19	and	5		24	20	and	1		21	20	and	20		40

The table of addition, like most other arithmetical tables, has a natural limit in the number 10, since from that number additions may be made in two or more columns. But a facility in reckoning may be acquired greatly beyond this limit; the present table, accordingly, is so extended as to afford ample scope for exercising the learner's memory, by a promiscuous examination in additions of frequent occurrence, but of somewhat difficult recollection, such as of 19 and 12, 13 and 17; to be preceded, of course, by additions still smaller. A recollection of the doubles of numbers, as of 14 and 14, will be found particularly useful. This method of examination from the table saves the printing of a thousand trifling questions. The learner is not expected to get the whole at once; he may take a column or more at a time; and here I would request him to recite all the tables in the book precisely as they are printed, without the addition or detraction of a single word. Thus, after the first *are* of a new number, he will repeat it no more, for it is ever after understood. Such a precision of method will produce both distinctness and despatch. When at a loss in addition, he will find it extremely useful to revert mentally to any tabular addition of the number in question which he does recollect, and rise to the number required by units.

Demonstration of rule.

When there is more than one column of figures, what will the sum of the right hand column consist of?—A right hand column of whole numbers consists of units, for these are always placed on the right; their sum therefore consists of units.

What is the largest number that can be set down in the place of units?—Nine are the most that can be set down

in the units' place ; because a unit more carries a number to the place of tens.

Should there be an excess of units above 9, on adding up a right hand column of figures, how may that excess be disposed of? — An excess of units above 9 from the right hand column must be carried to the nearest left hand column, for that is the place of tens; there will thus be no intermixture of tens with units.

To prevent such an intermixture, how may different sums be written down, in order to addition? — If units be placed under units, tens under tens, &c., this will keep the different values distinct.

What will any excess of tens consist of? — A single ten more than nine tens constitutes a hundred; therefore any excess of tens must consist of hundreds, and be carried to the column of hundreds.

Why? — Lest numbers of different value should be intermingled.

What would be the evil? — If intermingled, their true sum could not be obtained.

How does this appear? — Since there are only nine significant figures to express all numbers, unless their places be exactly kept, it will be impossible to know what numbers figures are meant to express.

Which column should first be added up? — As the excess of units must be added to tens, it would seem most convenient in addition to begin with units.

In how many ways may a column of figures be added together? — A column of figures may be added upward and downward.

May any advantage be derived from performing an addition in both ways? — The addition downward may detect errors committed in adding upward.

How may the detection arise? — Different figures occur in a different order, and different sums may be added to the same figures; so that the same errors are not likely to be committed in both manners of reckoning.

But since errors may be committed also in reckoning downward, what is to be done, in order to prove the correctness of an addition? — The columns must be reckoned up and down, till the results, in both manners, agree.

Can you now state the rule and the proof?

RULE OF ADDITION.

Write whatever numbers are to be added together one under another, exactly in due place, units under units, tens under tens, and so forth ; draw a line under the whole. Begin, from below, to add up the farthest column on the right. Under the units set down the units obtained, and carry the tens over to the column of tens: add up this column in like manner, note the tens below, and carry the hundreds ; so proceeding to the left hand column, under which set down its entire sum.

Proof: Make the addition downward, as it was made upward ; if the total be the same in both ways of reckoning, the addition is almost constantly right.

The proof of addition, by cutting off the top line, is trifling in itself, and impracticable in business.

APPLICATION.

94388	
950	
81067	
176405	Total.
121	

To apply the rule, let us consider the example in the margin. First observe the numbers, that they are arranged in due place, and exact order, according to the rule, units under units, tens under tens, hundreds under hundreds. We begin on the right, by saying, 7 and 8 are 15; fifteen consist of 5 units and 1 ten, because 10 and 5 are 15; we therefore set down 5 under the column of units, and carry the ten to the column of tens ; that is, we either mentally carry the ten, and say, 1 and 6 are 7 ; or, as in the example, we write a figure of 1 below all, under the column of tens, and then proceed with our addition, by saying, 1 and 6 are 7, and 5, twelve, and 8, twenty. Now 20, obtained from the column of tens, denote 20 tens, or 200; accordingly we write a 0 under that column, and carry 2 to the hundreds. We proceed: 2 and 9 are 11, and 3, fourteen; that is, 14 hundreds, because we are in the column of hundreds; but 1400 consist of 1 thousand and 4 hundreds ; we therefore continue: 1 and 1 are 2, and 4, six; namely, 6 thousands ; and there being no ten thousands, we conclude by adding up 8 and 9, which are 17. Now imagine yourself reciting this example to your instructor. While he has the book before him, you will read, from your slate, the lines of figures, from the topmost, thus: Ninety four thousand three hundred eighty eight. Nine hundred fifty. Eighty one thousand sixty seven. Total, one hundred seventy six thousand four hundred five. You will then proceed to add up the columns in the manner just directed, *and in the use of no more words.* Having made the addition upward, you will conclude by reciting the addition downward. Such a mode of recitation is almost of essential necessity to render you perfect in numeration, in which many advanced students are shamefully deficient.

Examples to be copied, added up, and recited.

1.	2.	3.	4.	5.
45	78	99	308	21
94	73	99	293	919
20	29	7	760	75
16	10	5	945	545
3	6	79	98	799
39	1	87	179	813

6.	7.	8.	9.
8038	5306	14	17086
4	7009	7566	93260
7600	196	6038	7
239	1973	4490	4033
5017	56	99	97591
4308	9809	1763	28040
1777	6545	228	63787

10.	11.	12.
6	515009	934718
294511	767981	570603
408763	7321	9
169321	900148	616721
587659	199357	38004
7033	8	998756
45617	793967	769789
88	249270	587609
890979	600798	678938

13.	14.	15.
7	5063295	5781164
6350840	2008	6183098
9279634	6400909	1010907
60010	200080	7430009
8643201	7531918	5007080
7600	4893717	776605
1956009	11	808083
6800098	365	7389
1476	8993695	7654321
9709999	7004070	9

16.	17.	18.
8	71459600	59812046
40356719	1703	37900031
71200901	32019546	53167700
18790560	67392001	70760607
60500937	90030979	7009
75055	55577766	49038195
236169	7036028	47
90050074	14000104	762020
36191017	576947	10989898
70007007	89765432	48170637

19.	20.	21.
9	291067036	756156716
876005063	507	680417
405786140	440405506	513209006
730570007	638781972	7980177
81045076	671003000	890043050
639958008	31117	888
700706070	109002	111805067
93	104176008	360746932
517102	779945109	856907008
871199340	900090008	570180917

22.	23.	24.
10	1604537391	5583970028
6300507167	7310004581	1672659432
5769190501	6900908009	7500010004
8032845787	7008156932	7496097210
5060819	3981607090	5037219003
3810007078	4201579081	83
164380001	7175	9143001998
7799888	2304590	4005167000
4410562891	9007913879	78567
7751560404	1748070067	8765432100

25.	26.	27.
11	33157006091	90096034218
17056300962	45558728	38101567020
90000490674	60708915047	33500091267
32816732199	23143069826	65932190505
46070161928	75679120090	84936
767	71281605066	19063716583
7006018	9	28769540000
51910188697	99360598073	200687958
20004763540	14004	33997617382
69891650099	829837674	47

28.	29.	30.
12	657891007037	498163959712
715607030561	381603781264	509631207058
763458977000	560739504123	618700780893
260043821989	712191000560	776542893718
900000000000	730819	831800056006
308	850503916782	909759412561
1456189	249999	315407135049
437610285033	879151063288	763515792984
609728853216	726035128795	820776504737
518162914935	500060831549	120000000000

Without large examples in addition, and the recitation of them, numeration will seldom be acquired, or at least retained. The objection, that such large numbers cannot be comprehended by a youth, seems to imply a non-perception of the uses that may be made of them.

SUBTRACTION.

Definition.

What is the third arithmetical operation? — The third operation in arithmetic is subtraction.

What is the meaning of the verb, to subtract? — To subtract is to take away.

How are things diminished? — Things are diminished by taking away part.

What is it to reduce to nothing? — To reduce a thing to nothing is to subtract, or take it away, altogether.

Can you give me an example? — If a man having five dollars only about him should lose them, this would be an accidental subtraction of the whole, and a reduction of his cash to nothing.

How might this be expressed in numbers? — This might be expressed in numbers, by subtracting 5 from 5.

Can you now define subtraction? — *Subtraction is the diminution of number, by taking equals from equals, or a less number from a greater.*

When the terms of a subtraction are equal, what is the result? — Equals subtracted from equals leave nothing.

Is this circumstance noted in arithmetic? — Either no mark is made, or a cipher must be set down.

What means the word none? — *None* signifies, not one.

When nought is taken from a number, what remains? — Nought from a number leaves that number, for nothing is taken from it.

Can a number be taken from nought, and leave any thing? — A number cannot be taken from nought; therefore a number from nought leaves none.

Can a greater number be taken from a less, and leave any thing? — A greater number can in part only be taken from a less; it can therefore leave none.

Can equals be subtracted, and leave any thing? — Equals from equals leave none.

Can you now recite the table of subtractions?

SUBTRACTION TABLE.

To be recited orderly till the learner is perfect, then promiscuously on examination.

1	from	1 leaves	0	1	from	15 leaves	14	2	from	7 leave	5
1	from	2	1	1	from	16	15	2	from	8	6
1	from	3	2	1	from	17	16	2	from	9	7
1	from	4	3	1	from	18	17	2	from	10	8
1	from	5	4	1	from	19	18	2	from	11	9
1	from	6	5	1	from	20	19	2	from	12	10
1	from	7	6	1	from	21	20	2	from	13	11
1	from	8	7					2	from	14	12
1	from	9	8					2	from	15	13
1	from	10	9	2	from	2 leave	0	2	from	16	14
1	from	11	10	2	from	3	1	2	rom	17	15
1	from	12	11	2	from	4	2	2	from	18	16
1	from	13	12	2	ftom	5	3	2	from	19	17
1	from	14	13	2	from	6	4	2	from	20	18

2	from	21	leave	19
2	from	22		20

3	from	3	leave	0
3	from	4		1
3	from	5		2
3	from	6		3
3	from	7		4
3	from	8		5
3	from	9		6
3	from	10		7
3	from	11		8
3	from	12		9
3	from	13		10
3	from	14		11
3	from	15		12
3	from	16		13
3	from	17		14
3	from	18		15
3	from	19		16
3	from	20		17
3	from	21		18
3	from	22		19
3	from	23		20

4	from	4	leave	0
4	from	5		1
4	from	6		2
4	from	7		3
4	from	8		4
4	from	9		5
4	from	10		6
4	from	11		7
4	from	12		8
4	from	13		9
4	from	14		10
4	from	15		11
4	from	16		12
4	from	17		13
4	from	18		14
4	from	19		15
4	from	20		16
4	from	21		17
4	from	22		18
4	from	23		19
4	from	24		20

5	from	5	leave	0
5	from	6	leave	1
5	from	7		2
5	from	8		3
5	from	9		4
5	from	10		5
5	from	11		6
5	from	12		7
5	from	13		8
5	from	14		9
5	from	15		10
5	from	16		11
5	from	17		12
5	from	18		13
5	from	19		14
5	from	20		15
5	from	21		16
5	from	22		17
5	from	23		18
5	from	24		19
5	from	25		20

6	from	6	leave	0
6	from	7		1
6	from	8		2
6	from	9		3
6	from	10		4
6	from	11		5
6	from	12		6
6	from	13		7
6	from	14		8
6	from	15		9
6	from	16		10
6	from	17		11
6	from	18		12
6	from	19		13
6	from	20		14
6	from	21		15
6	from	22		16
6	from	23		17
6	from	24		18
6	from	25		19
6	from	26		20

7	from	7	leave	0
7	from	8		1
7	from	9		2
7	from	10		3
7	from	11		4
7	from	12		5
7	from	13	leave	6
7	from	14		7
7	from	15		8
7	from	16		9
7	from	17		10
7	from	18		11
7	from	19		12
7	from	20		13
7	from	21		14
7	from	22		15
7	from	23		16
7	from	24		17
7	from	25		18
7	from	26		19
7	from	27		20

8	from	8	leave	0
8	from	9		1
8	from	10		2
8	from	11		3
8	from	12		4
8	from	13		5
8	from	14		6
8	from	15		7
8	from	16		8
8	from	17		9
8	from	18		10
8	from	19		11
8	from	20		12
8	from	21		13
8	from	22		14
8	from	23		15
8	from	24		16
8	from	25		17
8	from	26		18
8	from	27		19
8	from	28		20

9	from	9	leave	0
9	from	10		1
9	from	11		2
9	from	12		3
9	from	13		4
9	from	14		5
9	from	15		6
9	from	16		7
9	from	17		8
9	from	18		9
9	from	19		10

9	from	20	leave	11
9	from	21		12
9	from	22		13
9	from	23		14
9	from	24		15
9	from	25		16
9	from	26		17
9	from	27		18
9	from	28		19
9	from	29		20

10	from	10	leave	0
10	from	11		1
10	from	12		2
10	from	13		3
10	from	14		4
10	from	15		5
10	from	16		6
10	from	17		7
10	from	18		8
10	from	19		9
10	from	20		10
10	from	21		11
10	from	22		12
10	from	23		13
10	from	24		14
10	from	25		15
10	from	26		16
10	from	27		17
10	from	28		18
10	from	29		19
10	from	30		20

11	from	11	leave	0
11	from	12		1
11	from	13		2
11	from	14		3
11	from	15		4
11	from	16		5
11	from	17		6
11	from	18		7
11	from	19		8
11	from	20		9
11	from	21		10
11	from	22		11
11	from	23		12
11	from	24		13
11	from	25		14
11	from	26		15
11	from	27	leave	16
11	from	28		17
11	from	29		18
11	from	30		19
11	from	31		20

12	from	12	leave	0
12	from	13		1
12	from	14		2
12	from	15		3
12	from	16		4
12	from	17		5
12	from	18		6
12	from	19		7
12	from	20		8
12	from	21		9
12	from	22		10
12	from	23		11
12	from	24		12
12	from	25		13
12	from	26		14
12	from	27		15
12	from	28		16
12	from	29		17
12	from	30		18
12	from	31		19
12	from	32		20

13	from	13	leave	0
13	from	14		1
13	from	15		2
13	from	16		3
13	from	17		4
13	from	18		5
13	from	19		6
13	from	20		7
13	from	21		8
13	from	22		9
13	from	23		10
13	from	24		11
13	from	25		12
13	from	26		13
13	from	27		14
13	from	28		15
13	from	29		16
13	from	30		17
13	from	31		18
13	from	32		19
13	from	33		20

14	from	14	leave	0
14	from	15		1
14	from	16		2
14	from	17		3
14	from	18		4
14	from	19		5
14	from	20		6
14	from	21		7
14	from	22		8
14	from	23		9
14	from	24		10
14	from	25		11
14	from	26		12
14	from	27		13
14	from	28		14
14	from	29		15
14	from	30		16
14	from	31		17
14	from	32		18
14	from	33		19
14	from	34		20

15	from	15	leave	0
15	from	16		1
15	from	17		2
15	from	18		3
15	from	19		4
15	from	20		5
15	from	21		6
15	from	22		7
15	from	23		8
15	from	24		9
15	from	25		10
15	from	26		11
15	from	27		12
15	from	28		13
15	from	29		14
15	from	30		15
15	from	31		16
15	from	32		17
15	from	33		18
15	from	34		19
15	from	35		20

16	from	16	leave	0
16	from	17		1
16	from	18		2
16	from	19		3
16	from	20		4

16	from	21	leave	5
16	from	22		6
16	from	23		7
16	from	24		8
16	from	25		9
16	from	26		10
16	from	27		11
16	from	28		12
16	from	29		13
16	from	30		14
16	from	31		15
16	from	32		16
16	from	33		17
16	from	34		18
16	from	35		19
16	from	36		20

17	from	17	leave	0
17	from	18		1
17	from	19		2
17	from	20		3
17	from	21		4
17	from	22		5
17	from	23		6
17	from	24		7
17	from	25		8
17	from	26		9
17	from	27		10
17	from	28		11
17	from	29		12
17	from	30		13
17	from	31		14
17	from	32		15
17	from	33		16
17	from	34		17
17	from	35	leave	18
17	from	36		19
17	from	37		20

18	from	18	leave	0
18	from	19		1
18	from	20		2
18	from	21		3
18	from	22		4
18	from	23		5
18	from	24		6
18	from	25		7
18	from	26		8
18	from	27		9
18	from	28		10
18	from	29		11
18	from	30		12
18	from	31		13
18	from	32		14
18	from	33		15
18	from	34		16
18	from	35		17
18	from	36		18
18	from	37		19
18	from	38		20

19	from	19	leave	0
19	from	20		1
19	from	21		2
19	from	22		3
19	from	23		4
19	from	24		5
19	from	25		6
19	from	26		7
19	from	27	leave	8
19	from	28		9
19	from	29		10
19	from	30		11
19	from	31		12
19	from	32		13
19	from	33		14
19	from	34		15
19	from	35		16
19	from	36		17
19	from	37		18
19	from	38		19
19	from	39		20

20	from	20	leave	0
20	from	21		1
20	from	22		2
20	from	23		3
20	from	24		4
20	from	25		5
20	from	26		6
20	from	27		7
20	from	28		8
20	from	29		9
20	from	30		10
20	from	31		11
20	from	32		12
20	from	33		13
20	from	34		14
20	from	35		15
20	from	36		16
20	from	37		17
20	from	38		18
20	from	39		19
20	from	40		20

The table of subtraction has the same natural limit, in the number 10, with that of addition, because no figure in subtractor or minuend can exceed the 9; it is extended for the same purpose of exercising the learner's memory, by means of a promiscuous examination. It is recommended to him to use the very words that are printed, neither more nor fewer, in his recitations.

Terms.

What are the terms used in subtraction? — The terms in subtraction are, the minuend, the subtractor, and the difference.

Whence is the first? — Minuend is a Latin word abbreviat-

ed, signifying something to be diminished; it denotes the number to be subtracted from.

What the next — Subtractor is the number to be subtracted from the minuend.

This has hitherto been called the subtrahend; why does it seem desirable to propose a change? — Subtrahend is often mistaken for the number to be subtracted from, and subtractor would represent the number by which we operate.

What was the last mentioned term? — The *difference* is any number that remains after subtraction; hence it is also called the remainder.

Demonstration of Rule.

How would you set down two lines of figures, in order to addition? — Any two numbers that are to be added together, if set one under the other, must be written units under units, tens under tens; all exactly in due place.

Why? — The figures would otherwise be in confusion; and units, tens, hundreds, mingled together in the sum total.

Suppose one line is to be subtracted from another? — A number to be subtracted from some other I would arrange in the same manner, one under the other; fearing a similar confusion from an opposite method.

Which would be the most natural situation of the minuend? — The minuend should be placed above, that the operation may conveniently be carried below.

Where would you place the units of the difference? — The units of the difference should be placed under the units of the subtractor and minuend; for the same reason, of avoiding confusion.

Where would you begin? — I would begin on the right, as in addition, by subtracting the lowest numbers.

How would you begin? — I would subtract an equal from an equal, or a less number from a greater; for a greater number cannot be subtracted from a less.

Can you exemplify this? — We cannot take three dollars out of one.

You have learned the subtraction table, how do you proceed in subtracting 9 from 12? — In subtracting 9 from 12, I consider 12 as representing one heap of things, and 9 as so many things to be taken out of it.

Suppose 24 to represent two heaps, one of less than 9, how would you subtract 9, if directed first to take the smaller heap? — If 24 things were in two separate heaps, one of 4, the other of 20, I might first subtract the entire 4, and then take 5 from the 20, to make up 9.

What would these two heaps be called arithmetically? — In numbers, the 20 would be the heap of tens; the 4 would be the heap of units.

Arithmetically, could we take less than a whole ten from the heap of tens? — If the heap be represented by figures, we cannot take less than ten from the heap of tens.

Why not? — Because the place of tens cannot be filled by units; and if we should subtract less than a whole ten, there would be units remaining in the tens' place: this would create confusion.

Could you not follow this arithmetical manner, in subtracting from the heap itself? — I could subtract 10 from the heap of 20, add the 10 to the heap of 4; then subtract 9, leaving a small heap of 5.

What would the whole remainder be? — The whole remainder, from the two heaps would be 15.

Can you represent this actual subtraction in figures? — I can figure the 24 in three numbers, two of 10 each, one of 4; and from the two numbers making 14, I can subtract the 9; which leaving 5, I will set 5 in the units' place below.

$$\begin{array}{lrl} \text{equal to } 24 & \left\{\begin{array}{r}10\\ \left.\begin{array}{r}10\\ 4\end{array}\right\}\end{array}\right. & \Big\}\ \text{minuend}\\ & 9 & \text{subtractor}\\ \hline & 15 & \text{difference}\end{array}$$

Is this all the difference? — There still remains what represents the heap of 10, untouched; I will therefore set down a 1 in the tens' place of the difference, which will then be 15.

Suppose, for compactness' sake, we make of these three numerical heaps one; can you not still subtract 9? — From 24 I can subtract 9, by mentally taking a ten from the 20, adding it to the 4, and saying, 9 from 14, five.

$$\begin{array}{rl} 24 & \text{minuend}\\ 9 & \text{subtractor}\\ \hline 15 & \text{difference}\\ \hline\hline 1 & \end{array}$$

But having increased the 4 by 10, what ought you to do? — Certainly I ought to diminish the 20 by 10, if I suppose 10 taken from it.

How might this be done without altering figures? — This might be done, by carrying 1 to the tens' place, as in addition, and subtracting it, instead of adding.

Should we now add the difference to the subtractor, what sum would be made? — The difference added to the subtractor makes the minuend; for 15 and 9 are 24.

Is it so in every case? — It must be so in every case of subtraction; for the subtractor is taken from the minuend; and the difference, if any, shows how much less the subtractor is than the minuend.

What is the inference? — If the difference, added to the subtractor, make a sum equal to the minuend, the subtraction is right.

How then is subtraction proved? — Subtraction is proved by addition.

Can you now recite the rule and the proof?

RULE OF SUBTRACTION.

To subtract, write the minuend above and the subtractor beneath, units under units, &c. exactly in due place; draw a line under the two; subtract the lower right-hand figure, if equal or less, from that above it, noting any difference underneath the same column; if greater, add ten to the upper figure, subtract, set down the difference, carry one to the next lower place, subtract it thus increased, and so proceed to the left.

Proof. Add the difference to the subtractor; if the total be the same with the minuend, the subtraction is right.

APPLICATION.

907861	minuend
259450	subtractor
648411	difference
11	

Still farther to aid your comprehension of this rule, let us work an example together; namely, that in the margin. Suppose yourself reciting, you will begin thus: Nine hundred seven thousand eight hundred sixty one, the minuend. Two hundred fifty nine thousand four hundred fifty, the subtractor. Difference, six hundred forty eight thousand four hundred eleven. Nought from 1, one; 5 from 6, one; 4 from 8, four; 9 from 7, none; 9 from 17, eight; carry 1; 6 from 10, four; carry one; 3 from 9, six. Proof: 1 and 5, six; 4 and 4, eight; 8 and 9, seventeen; carry 1; 5 and 5, ten; carry 1; 7 and 2, nine. As you proceed in the proof, you notice, concerning every addition, whether the sum, or its lowest place, be the same with the figure above; for if it be not, you need proceed no farther in the proof, till you have again gone over the subtraction; that you may ascertain where the error lies, whether in the subtraction or addition. Be sure to use no more words in recitation than what I have attributed to you. To figure a proof in subtraction is superfluous, and unworkmanlike.

Examples to be wrought and recited.

1.	From	19220	subtract .	14110.	From	63712	subtract .	42701.
2.		74694	. .	23443.		49167	. .	38052.
3.		59769	. .	38727.		96957	. .	75133.
4.		10199	. .	9098.		27849	. .	6637.

5.	From 27828	subtract . 13506.	From 10378	subtract . 7056.
6.	10792	. . 6081.	10236	. . 3115.
7.	738930	. . 591046.	719456	. . 713490.
8.	119988	. . 103721.	600738	. . 470501.
9.	190267	. . 70385.	165080	. . 85427.
10.	700000	. . 651994.	200001	. . 190168.
11.	7100981	. 7085609.	1609074	. . 653450.
12.	5617007	. 5001008.	9111012	. . 8000901
13.	17003429	. 16078572.	70050014	. 38400509.
14.	12110012	. 5607384.	20100211	. 8769459.
15.	740012310	. 732160741.	832011230	. 395940817.
16.	110103252	. 713999.	291003321	. 270809612.
17.	603025421	. 49680756.	960102031	. 959800765.
18.	200200304	. 170393712.	103200150	. 86037659.
19.	9876543210	1234567890.	1234567890	. 76543219.
20.	7306005321	. 980193487.	1000000000	776830987.

Numeration, addition, and subtraction, being the three cardinal operations of arithmetic, varied in name and form only in every subsequent process, it will be useless for the learner to proceed, who is not well acquainted with the definition and rules we have hitherto been considering. To such a one it will be no loss of time, but a saving, should he recommence the whole subject anew. The next thing to be learned is the symbolic abbreviations; printed to front the first page. Multiplication succeeds to it.

MULTIPLICATION.

What is the symbol of multiplication?

Of what part of speech are numbers? — Numbers are nouns adjective.

Why so ranked? — Because they are expressive of quality, of individuality, namely; and we speak of one thing, two things, as we say a good thing, good things.

Is there any distinction among numerical adjectives? — Numbers are of two kinds, cardinal and ordinal.

What are the former sort? — The cardinal numbers are one, two, three, &c. They are called cardinal, or principal, because by them principally we compute.

What are the latter sort? — The ordinal numbers are first, second third, &c., denoting the order of occurrence; as the fourth operation of the first part of arithmetic, which we are now upon.

Are numerical adjectives ever used alone? — Numbers are

as frequently used alone as otherwise, their substantives being understood.

Can you exemplify it? — No one goes thither; the first that departed.

Do adjectives ever receive a plural termination? — Adjectives are sometimes treated as substantives, by the annexation of articles and the plural termination; as a good, goods; a dozen, dozens.

Definition.

What means the verb to multiply? — To multiply is to increase.

How are things increased? — Things are increased by additions made to them.

How would you learn the amount of five bags of dollars, containing 120 each? — To find the amount of 5 times 120 dollars I should either set that number down 5 times in column, and add them up; or I might say to myself, 120 and 120 are 240; two bags; 240 and 240 are 480; four bags; 480 added to 120 more make, in all, 600.

Addition.

$$\begin{array}{r} 120 \\ 120 \\ 120 \\ 120 \\ 120 \\ \hline 600 \end{array}$$

This would be addition; could you compute the sum thus: 5 twelves are 60; annex a cipher, and the amount is 600; what would you say of multiplication? — Multiplication is a short mode of addition.

Multiplication.

$$\begin{array}{r} 120 \\ 5 \\ \hline 600 \end{array}$$

What is a number taken two times? — A number taken two times is twice that number.

If 5 be added to 5, how may the operation be described by the use of a personal pronoun? — The same number twice taken may be called a number added into itself.

What is a number repeated? — To repeat a number is also to add it into itself, if the repetitions go to form a single sum.

Are these terms all of the same signification? — To take so many times, to repeat, and to add into itself, are terms signifying the same thing in multiplication.

How often may a number be taken? — A number may be taken as many times as shall be requisite; once or oftener.

Can a number be taken part of a time? — Part of a number can be taken; as a half.

Do you recollect what every number was said to be made up of? — Every number is made up of units or parts of a unit.

Can you now define multiplication? — *Multiplication is a*

short mode of adding a number into itself, as many times, and to the extent of as many parts, as there are units and parts of a unit in the same or some other number. Multiplication also denotes the taking of a number once, or in part only.

A little work has lately been published in Boston, called "The Ready Multiplier"; it exhibits a very ingenious discovery of a principle peculiar to factors, consisting of two places, *and having the same left hand digit.* This latter limitation, of course, renders the principle useless in eight cases out of nine under the former limitation; and the old methods of practice in the table remain as useful as ever. Peculiar devices of this sort are seldom of the least efficiency. Recollection, unaided by general principles, is, at the most, but a recollection of particulars; this will carry us but a short range in multiplication; and the only question is, how far that range can beneficially, and *practically*, be extended. The present author has attempted a slight extension of that range, and not without reasons assigned for it; should the attempt succeed, he will have done something for the business of life.

Can you recite the multiplication table?

MULTIPLICATION TABLE.

To be recited orderly, till the learner is perfect; then promiscuously, on examination.

Once nought is none, &c.

Twice	nought	are	0
Twice	1		2
Twice	2		4
Twice	3		6
Twice	4		8
Twice	5		10
Twice	6		12
Twice	7		14
Twice	8		16
Twice	9		18
Twice	10		20
Twice	11		22
Twice	12		24
Twice	13		26

Thrice	nought	are	0
Thrice	1		3
Thrice	2		6
Thrice	3		9
Thrice	4		12
Thrice	5		15
Thrice	6		18
Thrice	7		21
Thrice	8	are	24
Thrice	9		27
Thrice	10		30
Thrice	11		33
Thrice	12		36
Thrice	13		39

4	noughts	are	0
4	1s		4
4	2s		8
4	3s		12
4	4s		16
4	5s		20
4	6s		24
4	7s		28
4	8s		32
4	9s		36
4	10s		40
4	11s		44
4	12s		48
4	13s		52

5	noughts	are	0
5	1s		5
5	2s		10
5	3s		15
5	4s		20
5	5s		25
5	6s		30
5	7s		35
5	8s		40
5	9s		45
5	10s		50
5	11s		55
5	12s		60
5	13s		65

6	noughts	are	0
6	1s		6
6	2s		12
6	3s		18
6	4s		24
6	5s		30
6	6s		36
6	7s		42

6	8s are	48
6	9s	54
6	10s	60
6	11s	66
6	12s	72
6	13s	78

7	noughts are	0
7	1s	7
7	2s	14
7	3s	21
7	4s	28
7	5s	35
7	6s	42
7	7s	49
7	8s	56
7	9s	63
7	10s	70
7	11s	77
7	12s	84
7	13s	91

8	noughts are	0
8	1s	8
8	2s	16
8	3s	24
8	4s	32
8	5s	40
8	6s	48
8	7s	56
8	8s	64
8	9s	72
8	10s	80
8	11s	88
8	12s	96
8	13s	104

9	noughts are	0
9	1s	9
9	2s	18
9	3s	27
9	4s	36
9	5s	45
9	6s	54
9	7s	63
9	8s	72
9	9s	81
9	10s	90
9	11s	99
9	12s	108
9	13s	117

10	noughts are	0
10	1s	10
10	2s	20
10	3s	30
10	4s	40
10	5s	50
10	6s	60
10	7s	70
10	8s	80
10	9s	90
10	10s	00
10	11s	110
10	12s	120
10	13s	130

11	noughts are	0
11	1s	11
11	2s	22
11	3s	33
11	4s	44
11	5s	55
11	6s	66
11	7s are	77
11	8s	88
11	9s	99
11	10s	110
11	11s	121
11	12s	132
11	13s	143

12	noughts are	0
12	1s	12
12	2s	24
12	3s	36
12	4s	48
12	5s	60
12	6s	72
12	7s	84
12	8s	96
12	9s	108
12	10s	120
12	11s	132
12	12s	144
12	13s	156

13	noughts are	0
13	1s	13
13	2s	26
13	3s	39
13	4s	52
13	5s	65
13	6s	78
13	7s	91
13	8s	104
13	9s	117
13	10s	130
13	11s	143
13	12s	156
13	13s	169

Submultiples: Twice 7 are 14. Thrice 5 are 15. Twice 8 are 16. Thrice 6 are 18.

What are multiples? — Multiples are the product of two or more numbers; as 14 of 2 and 7.

What are submultiples? — Submultiples are the numbers producing a multiple; as 3 and 5 produce 15.

The multiplication table, invented by the illustrious Pythagoras, stopped at its natural limit, the number 10; the English school, which retains many of the duodecimal divisions of the ancient northern na-

tions, proceeds to 12; to the author it has appeared, that 13 would afford, if not a more natural, at least a more advantageous, limit beyond 10; because the number 13 is immediately succeeded by three multiples, best operated with, when alone, by their submultiples; the aid of memory would thus virtually be carried to more than one half additional of the ancient tabular factors, with the additional burden only in the present instance, of a single factor and its products. Still, should the innovation be objected to, the 13s may be altogether neglected. Yet, the author cannot help thinking, to no releasement of the pupil; for the burden will be found very light, and the advantage great in practice, at the same time that a retention of the 13s affords scope for promiscuous examination in the table. Such an examination, it is hoped, will be frequent and thorough; the matter is of the greatest importance, and of daily necessity; and a score of pages of questions, carrying the pupil not a whit farther, will be saved by it. The questions concerning multiples should follow every recitation of the table. The language adopted in the table, though not everywhere perfectly grammatical, is purposely designed to promote distinctness, and to save time, in recitation; for it is an intolerable bore, to hear children, at every step in multiplication, drawling out their 2 *times* 2 *is four*, &c.

Multiplication Table,

exhibiting, at a single view, the squares, and the products, of numbers from 1 to 13.

1	2	3	4	5	6	7	8	9	10	11	12	13
2	4	6	8	10	12	14	16	18	20	22	24	26
3	6	9	12	15	18	21	24	27	30	33	36	39
4	8	12	16	20	24	28	32	36	40	44	48	52
5	10	15	20	25	30	35	40	45	50	55	60	65
6	12	18	24	30	36	42	48	54	60	66	72	78
7	14	21	28	35	42	49	56	63	70	77	84	91
8	16	24	32	40	48	56	64	72	80	88	96	104
9	18	27	36	45	54	63	72	81	90	99	108	117
10	20	30	40	50	60	70	80	90	100	110	120	130
11	22	33	44	55	66	77	88	99	110	121	132	143
12	24	36	48	60	72	84	96	108	120	132	144	156
13	26	39	52	65	78	91	104	117	130	143	156	169

Terms.

What terms are used in the fourth operation? — The terms used in multiplication are multiplicand, multiplier, product; partial, and separate, product.

Whence the first name? — Multiplicand is a Latin word abbreviated, signifying something to be multiplied.

Have the two first terms no other name? — Multiplicand and multiplier are also called factors, because they effect the operation.

What are the last named? — A product is the sum produced by the act of multiplication; entire product is the sum total of separate products; partial product is the sum produced by every particular act of multiplication in a series.

Demonstration of Rule.

What is the product of 4 fives? — Four 5s are 20.

What is the product of 5 fours? — Five 4s are 20 also.

Does reversing the order of the factors make any change in the result? — It makes not the slightest difference in the product which factor is made the multiplicand.

Why not? — Because either way both factors multiply each other.

Can you show this from the example? — In multiplying 4 into 5, five are taken 4 times, and four are taken 5 times.

What is the different meaning of multiplying into, and multiplying by? — *Multiplying into* makes no distinction of the multiplier; *multiplying by* points it out.

Of the factors 19 and 7, which would you choose for a multiplier? — In multiplying 19 into 7, I should make 7 the multiplier; because it is easier to multiply by one figure than two, and I have learned to multiply by 7.

How would you multiply 19 into 30? — I should make 30 the multiplier of 19; because in 30 there is only one significant.

Would not the cipher require some additional work? — The cipher, being nothing, could multiply nothing.

But does it not give additional value? — The cipher could only affect the place of the significants in the product, by showing them to be the product of three tens, not of three units.

In what manner would this appear? — It would appear by the annexation of a cipher on the right of the product.

```
 19 multiplicand
 30 multiplier
---
570
---
2
```

Of factors then, which would you choose for the multiplicand? — I would choose that factor for the multiplicand which has the greatest number of significants.

Should there be ciphers on the right of the multiplier? — Ciphers on the right of the multiplier I would also annex on the right of the product.

Suppose ciphers on the right of both factors? — Any ciphers on the right of the factors I would annex on the right of the product; for each must denote an additional place.

What would you do with ciphers between significants? — Ciphers between significants must be passed over in the factors, for they can multiply nothing; although they must, in some manner, affect the significants of the product.

Where should the multiplicand be placed in an operation? — The multiplicand should be placed above, that the working part may extend below.

How should the digits of the multiplier be arranged? — In addition and subtraction units must be placed under units.

Can you tell me why? — In addition, units must be added to units; in subtraction, units must be taken from units.

What may be perceived from ciphers on the right of factors? — From ciphers on the right of factors we see at once how many places must be annexed to the significants of the product.

What produces those significants? — The significants of a product can be produced only by the significants of the factors.

Then what would be a convenient arrangement of the factors? — It would be convenient for multiplying to set the right hand significants of the factors, one under the other.

Would it create any confusion? — This would cause no confusion, for the product of the farthest significants would be set down farthest; and we know whether any, or how many, ciphers are to be annexed to the right of the entire product.

$$\begin{array}{r} 140 \\ 300 \\ \hline 42000 \end{array}$$

Arrangement of products.

Where would you begin to multiply? — I would begin by multiplying the two right hand significants into each other.

What would you do with the first partial product? — I would set down the units and carry the tens, as in addition.

How would you proceed? — I would multiply the tens, and carry the hundreds; and so all through.

Should there be any other significant in the multiplier, what would be your course? — I would proceed with the next significant of the multiplier in like manner.

$$\begin{array}{r} 294 \\ 4 \\ \hline 1176 \end{array}$$

A cipher might intervene? — An intervening cipher I should neglect, and proceed with the next significant.

What would be the value of the next significant? — The next significant would be increased ten times in value, for any intervening cipher on its right.

$$\begin{array}{r} 205 \\ 103 \\ \hline 615 \\ 205 \\ \hline 21115 \end{array}$$

How would you exhibit this in the product? — An increased value in any significant of the multiplier, might be made to appear in the product, by a corresponding advance of its first partial product toward the left.

What is the product of units multiplied by tens? — The product of units multiplied into tens cannot be less than tens; for a single unit taken ten times is equal to ten.

What then must be the value of every first partial product? — The product of no two integers can be of less value than either factor; for each is added into itself once or oftener.

Can it be greater? — It may be greater, but the lowest place of a product is the same with that of the multiplier; because it cannot be lower, if the factors are whole numbers; and any higher place will require the noting of other places additional, in the product.

Can you exemplify this? — Once 9, gives 9 to the units' place of the product; twice 9 give 8 to the units' place of the product; twice 10 give a cipher to the same units' place.

How is a sum total obtained? — A sum total is obtained by addition.

What are the steps in multiplication? — Every particular multiplication has its partial products, and separate product, if there be more than one.

How must we obtain an entire product? — When there are separate products, they must be added together, to make up the entire product.

Where will you note the lowest place of every first partial product? — The lowest place of every first partial product should be noted under the significant, exactly, by which it was produced.

Wherefore? — Because the lowest place of every first partial product is that of the multiplier; and as separate products are to be added together, every figure will thus fall into its right place.

What will you do in respect of any ciphers on the right of factors? — To whatever part of the entire product is obtained by the significants of the factors, as many ciphers must be annexed as there may be on the right of both factors.

Should there be ciphers on the right of factors, will the first partial product be correctly placed, if arranged as you describe? — Since the lowest place of every product, in the multiplication of integers, is the same with that of the multiplier, it cannot be incorrectly placed, if set underneath it exactly; provided the right hand significant of each factor also line one with the other.

Should ciphers on the right be allowed to interfere with the significants of a factor? — Ciphers on the right of a factor, placed under significants, create the hazard of a defective product through oversight, by a non-annexation to the right.

```
             221             221
             520              520
            ----             ----
             442             442
           1105            1105
           -----           ------
defective  11492 product   114920 entire product
```

Fractional Factors.

When fractions make part of a number, where are they placed? — Fractions are always placed on the right of whole numbers; for being lowest in value, their place is the lowest.

With fractions on the right, how are factors to be arranged? — ***Common fractions in one factor must extend beyond the right of the other factor, whatever be its digits on the right, significant or ciphers.***

Why? — Because if intermingled with whole numbers, they fill the place of units; their lowest integral product will, of course, be in the place of units; consequently, every first partial product being placed under its multiplier, there will be two separate products, one by fractions, another by units, whose farthest integral places are of the same value, set one to the left of the other.

```
  208
   7¼
  ---
   52
 1456
 -----
 14612 excessive p.
```

With what further result? — Any other separate product will also be advanced from the same cause, and the final product be greatly in excess.

```
  208
    7¼
  ---
   52
 1456
 -----
 1508 entire pro.
```

Does the same requirement follow from any other maxim we have laid down? —

To place common fractions under significants, defeats the maxim, that the right hand significants of integral factors should exactly correspond.

Suppose there should be fractions in each factor? — With common fractions in each factor there can be no difficulty, since they will naturally be placed one underneath the other.

2080	
$7\frac{1}{4}$	
520	
14560	
15080	entire pro.

Multiplication by powers of 10.

Twelve multiplied into 1 produce how many? — Twelve 1s are 12.

Twelve multiplied into 10 produce how many? — Twelve 10s are 120.

If 12 be taken 100 times, what is the product? — Twelve, multiplied into 100, become 1200.

Multiplication into 1 produces what figures? — Multiplication into 1 repeats the figures of the other factor.

What more does multiplication into 10 or 100? — Multiplication into 10, or 100, also repeats the figures of the other factor, annexing a cipher or more to the product.

What then is multiplication by powers of 10, as we say? — Multiplication into 10, or a unit with any number of ciphers, advances the other factor one place to the left for every cipher; and nothing more.

Arithmetical multiplication not always addition.

If I take a number one time do I increase it? — A number taken once remains the same.

Taken once and a half, is it increased? — A number taken once and a half is increased one half, or a third of the entire product; as a dollar, taken once and a half, is increased from two half dollars to three half dollars.

What do you say of a number multiplied to the extent of a half? — A number multiplied to the extent only of one half must be diminished; if, multiplied once, it be not increased.

A half is a fraction; what then is the effect of multiplication into fractions? — Multiplication into a fraction produces a number less than some one of the factors; for of some one, at least, a part only is taken.

In multiplying whole numbers into fractions, are the fractional factors diminished? — Fractional factors cannot be

diminished in the product by integral factors; for taken once only, they continue the same.

Are they ever increased?—If an integral factor be more than a unit, the fractional factor is necessarily exceeded in the product; for it is taken more than once; so half a dollar, taken twice, produces a dollar.

Let it be taken in part only, what is the product?—Half a dollar taken only in part must produce less than half a dollar.

Hence what may be inferred concerning the multiplication of fractions into fractions?—Fractions multiplied into fractions produce less than either factor; for a part only of each factor is taken.

Does multiplication now appear to be always a mode of addition?—Multiplication cannot be addition when the multiplier is a unit or a fraction; for addition is increase; and things taken once, or in part only, are not increased.

But if the multiplier be increased, is there not addition?—Even the multiplier is diminished when both factors are fractions; but the increase of no multiplier, unless it be the true multiplicand, affects our consideration of the product; for the true multiplicand is always of the kind taken in the product.

What then does multiplication imply?—Arithmetical multiplication has two meanings, one of repeating, or addition; the other, simply of taking, in whole or in part.

Submultiples.

What are tabular numbers?—Tabular numbers are factors within the multiplication table; namely, from 1 to 13.

In multiplying by 18, may not the process be abridged by the use of certain tabular factors?—Thrice 6 are 18; therefore if the multiplicand be first taken 3 times, and its product be taken 6 times, the last product will be the same with the entire product to be obtained by the factor, 18.

What abbreviation would this be?—This would render addition unnecessary; for in multiplying by 18, there must be separate products, since 18 is not a tabular number, its products therefore not recollected.

Why should the last and the entire product be the same in these cases?—When a single multiplier is used, there is but one multiplicand; when a product is made a successive multiplicand, the last multiplier being as much diminished as the last multiplicand is increased, the last product will, of course, be the same with the entire product of the original multiplicand.

$$\begin{array}{r} 930\times18 \qquad \\ 3\times6=18 \\ \hline 2790 \qquad \\ 6 \qquad \\ \hline 16740 \qquad \\ \hline\hline \end{array}$$

How do you exemplify this? — If the multiplier be 18, and the multiplicand be, by any process, increased 3 times, the multiplier itself must be diminished 3 times; or only a third part of it made the factor; such a diminution reduces it to 6; for 6 are a third part of 18.

What are such diminished multipliers called? — Submultiple is the name given to factors producing a multiple number.

930
18
744
93
16740

Why so called? — Submultiples are equivalent to subordinate multipliers.

Can you now recite the rule?

RULE OF MULTIPLICATION.

To multiply, set down the factor most suitable for the multiplicand, and underneath it the multiplier, the right hand significant of the one under the corresponding significant of the other, ciphers on the right of either, and common fractions, if any, extending beyond; draw a line under the two factors. Multiply the right hand significant below into that above, set down the units of the product, carry the tens to the next product, and so proceed to the left. In like manner multiply every other significant there may be of the multiplier into the multiplicand, beginning as before, and placing the right hand figure of every first partial product in the same column exactly with that significant of the multiplier by which it was produced. Draw a line under the whole; add up the separate products, if any, and set down on the right any ciphers there may be on the right of the factors; the sum total is the entire product, if the factors contain no common fraction; if otherwise, make the product by any fractions the first separate product or products.

Should there be a multiple factor, the product of tabular numbers, take the multiplicand by a submultiple, their product by another, and so to the last; the last product will be the entire product.

Proof of multiplication is by division. (See Multiplication and Division.)

A mode of proving multiplication, by casting out the 9s, as it is called, need not here be discussed. Constituting an additional process, perhaps saving no time, supposing a knowledge of division before it has been taught, and, above all, uncertain in itself, it may well be considered entirely out of place in a system of practical accounts. The subject of proof will be resumed under the head referred to. What has been said of fractions, is unavoidably introduced; the author

could not content himself with giving imperfect rules, to retrace his after steps, in order to remedy defects: he prefers anticipating the pupil's knowledge, to leading him into error.

APPLICATION.

After the examples already given, a very few will suffice for farther illustration. Let the scholar recite the example in the margin thus: Forty eight thousand seventy five, the multiplicand; twelve, the multiplier; five hundred seventy six thousand nine hundred, the product. Twelve 5s, 60; nought, and carry 6; twelve 7s, 84, 90; set down 90; twelve 8s, 96; 6, and carry 9; twelve 4s, 48, 57. There is not the slightest occasion for uttering a single word more than what is here written. It is not necessary to say, twelve 7s *are* 84, *and* 6 *are* 90; because the words italicized are understood, and because a youth should accustom himself to despatch in reckoning and recitation, though not to the sacrifice of clearness and distinctness; but this concise manner of expression will aid both.

1st.

```
 48075 multiplicand
    12 multiplier
 ------
576900 product
 ======
   9 6
```

Recite the next example from the margin: Seven hundred sixty one thousand ninety eight, the multiplicand; thirty nine, the multiplier; 13 and 3, submultiples of 39; nine million eight hundred ninety four thousand two hundred seventy four, the first product; twenty nine million six hundred eighty two thousand eight hundred twenty two, the last product. Thirteen 8s, 104; 4 and carry 10; thirteen 9s, 117, 127; 7, and carry 12; thirteen noughts, none; 2, carry 1 (namely from the 12); thirteen 1s, 13, 14; 4, and carry 1; thirteen 6s, 78, 79; 9, and carry 7; thirteen 7s, 91, 98. It will be perceived from the example, and the mode of recitation given, that the tens must be carried, in the additions to the several partial products made, mentally. In like manner should the last product be recited; then the larger examples that follow; with the prior *enumeration*, among other terms, of every separate product.

2d.

```
761098×39
    13×3=39
 --------
  9894274
        3
 --------
 29682822
 ========
```

```
   3d.              4th.             5th.

   61983           848500
     625           709200            490
 --------         -------             57
  309915           16970            ----
 123966           76365              343
371898           59395              245
 --------     ------------          ----
38739375      601756200000          27930
========      ============          =====
```

The different distances of the first partial products from the

place of units may be supposed filled with ciphers. In the third example, for instance, 2 multiplied into 3, is a multiplication of 20 into 3, because the 2 is in the place of tens; the product is therefore 60; but the place of units being already marked by the first separate product, it is unnecessary to note any place to the right of that immediately under the succeeding significant multiplier itself. When the multiplicand only has a cipher on the right, it is customary to set the units of the multiplier underneath it, and to insert it in every separate product; the necessity of so doing is not very obvious.

Examples to be wrought and recited.

What is the product of

1.	4567×5?	98421×7?	14678×3?	2718×4?
2.	9×3068?	6×49420?	8×73014?	2×9998?
3.	45912×12?	306907×11?	910078×13?	4×68301?
4.	13×1210523?	9×5059202?	10×498610708?	5380176917×14?
5.	8×903145007?	671408200×7?	9×500999210?	15×3841007061?
6.	2501019667×11?	718024978×18?	13×45177289?	21×139080564?
7.	12×9966300176?	100×500880324?	27×906799521?	932491671×12?
8.	14067324×39?	378044097×67?	477955607×96?	80015868310×83?
9.	86×3701560028?	71×1741430555?	94×63979588?	110×4348050176?
10.	303591608×34?	595730832×49?	62×17950681?	108×619098077?
11.	90×438544689?	70×736004519?	600×28604178?	84900290×506?
12.	820×763580?	650×6696770?	5800×7914860?	268340700×960?
13.	990×110903516?	308413990×6010?	56855190×7080?	14400×759841200?

DIVISION.

Definition.

What is the fifth arithmetical operation? — The fifth operation in arithmetic is division.

What are the symbols of division?

What is it to divide? — To divide is to sever a part from a whole, or a whole into equal parts; the former process is usually denominated subtraction.

Can you exemplify the latter? — Ten dollars may be divided into four sums of two dollars and a half each.

Has divide any other meaning? — To divide is also to distribute.

How would you distinguish between the verbs sever and divide? — To sever is to part, equally or unequally; to divide is so to part as to make an equal distribution.

How is distribution equalized? — To distribute equally, we consider how many turns the thing for distribution will go among the sharers.

Can you give me an example? — If four dollars are to be distributed between two men, we instantly perceive that two dollars are the share of each; because in distributing by pairs, one turn will exhaust the four; by single dollars, two turns.

Are not turns somewhat differently denominated? — In arithmetic, turns are usually called times; because different turns are made at different times.

Then what is part of a time? — Part of a turn is often called part of a time.

Can you exemplify part of a turn? — Five dollars, for distribution among four men will go one turn of a dollar to each, and the fourth part of another turn; because there will remain one dollar undistributed.

May not the dollar be changed? — By changing the dollar, each of the four men may receive a quarter additional; this turn however, being of diminished value, may, in comparison, be called part of a turn.

Suppose as many coppers to be distributed, what will be the procedure? — Five cents among four persons would go one turn to each, leaving a cent undistributed, or distributable, at the most, between two; for half cents are coined: this would strictly be going only part of a turn.

How is the act of distribution performed? — In effecting an equal distribution, we subtract every required part from the whole, till it be exhausted, or less remain than will go a single turn.

How is the required part known? — The number to be subtracted is expressed either as a part numbered or valued.

Can you exemplify the first case? — If the question be, what is the value of a fifth part of fifteen dollars, the subtractor, [or divisor] 5, expresses number only; for whatever value may attach to a 5th part is made the subject of the answer.

The second case? — If it be inquired, how often five dollars are contained in fifteen, the subtractor, [or divisor] 5, expresses value; for whatever value 15 represent, it is a number of turns only that is demanded, and which the answer must give.

What would be the process in the first case? — If the subtractor be a part numbered, as the 5th part of 15 dollars, I must find how often 5 can be subtracted from 15; for the subtraction of every 5, will give 1 dollar to every 5th part.

In the second case? — If the subtractor be a part valued, as a portion of 5 dollars, in 15 to be so apportioned, I must still find how often 5 can be subtracted from 15; because

every such subtraction denotes 1 turn of the entire value of 5 dollars; and the number of turns is the thing demanded.

How would you make these subtractions, and note them? — To learn how many times 5 can be subtracted from 15, I might say, 5 from 15 leave 10; one five; 5 from 10 leave 5; two fives; 5 from 5, none; in all, three turns of five each.

Could you say at once, by division, 5s in 15, three; what would be the inference? — Division would thus appear to be a short mode of subtraction.

How then would you divide a number into parts? — To divide a number into parts of a value already fixed, the number of turns must be found which the value of each part will go in the whole; for the number of turns is the number of the parts.

How will you value a part? — To value each in a number of parts specified, the turns must be found which the whole number of parts will go in the sum to be parted; for every complete turn of the whole number gives the value of a unit to every single part.

In what does division differ from subtraction? — Subtraction is a single operation, whereas in division we also employ multiplication; in subtraction, we subtract once only, and when nothing remains, nothing is noted; but in division, every subtraction is noted, as one or more turns, and a remainder, as part of a turn.

Why is this? — The object of division is to find the number of turns that one number will go in another; this is accomplished by means of subtraction; therefore every subtraction must be noted, or nothing will be found.

In one hundred cents how many dollars? — A hundred cents are equal to a single dollar only.

Which then is of the highest value? — A cent is of a much lower value than a dollar.

Substitute dollars for cents; how will their numbers be affected? — By substituting dollars for cents, numbers will become a hundred times less.

How are numbers lessened, and the diminution noted? — Numbers are lessened by subtraction, and their diminution is noted by division.

Can you now define division? — *Division is the mode of subtracting a number, once, oftener, or in part, for the distribution of numbers into equal parts, for the valuing of a known part, and for the raising of lower values to higher.*

Have you learned the table of divisions?

DIVISION TABLE.

To be recited orderly till the learner is perfect, then promiscuously on examination.

2s in 1 are $\frac{1}{2}$	5s in 4 are $\frac{4}{5}$	8s in 7 are $\frac{7}{8}$	11s in 10 are $\frac{10}{11}$
2s : 2 = 1	5s : 5 = 1	8s : 8 = 1	11s : 11 = 1
2s : 4 = 2	5s : 10 = 2	8s : 16 = 2	11s : 22 = 2
2s : 6 = 3	5s : 15 = 3	8s : 24 = 3	11s : 33 = 3
2s : 8 = 4	5s : 20 = 4	8s : 32 = 4	11s : 44 = 4
2s : 10 = 5	5s : 25 = 5	8s : 40 = 5	11s : 55 = 5
2s : 12 = 6	5s : 30 = 6	8s : 48 = 6	11s : 66 = 6
2s : 14 = 7	5s : 35 = 7	8s : 56 = 7	11s : 77 = 7
2s : 16 = 8	5s : 40 = 8	8s : 64 = 8	11s : 88 = 8
2s : 18 = 9	5s : 45 = 9	8s : 72 = 9	11s : 99 = 9
2s : 20 = 10	5s : 50 = 10	8s : 80 = 10	11s : 110 = 10
2s : 22 = 11	5s : 55 = 11	8s : 88 = 11	11s : 121 = 11
2s : 24 = 12	5s : 60 = 12	8s : 96 = 12	11s : 132 = 12
2s : 26 = 13	5s : 65 = 13	8s : 104 = 13	11s : 143 = 13

3s in 2 are $\frac{2}{3}$	6s in 5 are $\frac{5}{6}$	9s in 8 are $\frac{8}{9}$	12s in 11 are $\frac{11}{12}$
3s : 3 = 1	6s : 6 = 1	9s : 9 = 1	12s : 12 = 1
3s : 6 = 2	6s : 12 = 2	9s : 18 = 2	12s : 24 = 2
3s : 9 = 3	6s : 18 = 3	9s : 27 = 3	12s : 36 = 3
3s : 12 = 4	6s : 24 = 4	9s : 36 = 4	12s : 48 = 4
3s : 15 = 5	6s : 30 = 5	9s : 45 = 5	12s : 60 = 5
3s : 18 = 6	6s : 36 = 6	9s : 54 = 6	12s : 72 = 6
3s : 21 = 7	6s : 42 = 7	9s : 63 = 7	12s : 84 = 7
3s : 24 = 8	6s : 48 = 8	9s : 72 = 8	12s : 96 = 8
3s : 27 = 9	6s : 54 = 9	9s : 81 = 9	12s : 108 = 9
3s : 30 = 10	6s : 60 = 10	9s : 90 = 10	12s : 120 = 10
3s : 33 = 11	6s : 66 = 11	9s : 99 = 11	12s : 132 = 11
3s : 36 = 12	6s : 72 = 12	9s : 108 = 12	12s : 144 = 12
3s : 39 = 13	6s : 78 = 13	9s : 117 = 13	12s : 156 = 13

4s in 3 are $\frac{3}{4}$	7s in 6 are $\frac{6}{7}$	10s in 9 are $\frac{9}{10}$	13s in 12 are $\frac{12}{13}$
4s : 4 = 1	7s : 7 = 1	10s : 10 = 1	13s : 13 = 1
4s : 8 = 2	7s : 14 = 2	10s : 20 = 2	13s : 26 = 2
4s : 12 = 3	7s : 21 = 3	10s : 30 = 3	13s : 39 = 3
4s : 16 = 4	7s : 28 = 4	10s : 40 = 4	13s : 52 = 4
4s : 20 = 5	7s : 35 = 5	10s : 50 = 5	13s : 65 = 5
4s : 24 = 6	7s : 42 = 6	10s : 60 = 6	13s : 78 = 6
4s : 28 = 7	7s : 49 = 7	10s : 70 = 7	13s : 91 = 7
4s : 32 = 8	7s : 56 = 8	10s : 80 = 8	13s : 104 = 8
4s : 36 = 9	7s : 63 = 9	10s : 90 = 9	13s : 117 = 9
4s : 40 = 10	7s : 70 = 10	10s : 100 = 10	13s : 130 = 10
4s : 44 = 11	7s : 77 = 11	10s : 110 = 11	13s : 143 = 11
4s : 48 = 12	7s : 84 = 12	10s : 120 = 12	13s : 156 = 12
4s : 52 = 13	7s : 91 = 13	10s : 130 = 13	13s : 169 = 13

The table of divisions is the solution of the multiplication table, disentangling, as it were, what the latter involves. In promiscuous examinations of the pupil, it is recommended, not to confine the questions to multiples, but to extend them to numbers intervening, that he may be familiarized, as speedily as possible, to the expression of fractions. The first quotient of every new digit in the preceding table is of this kind, to serve as an example. The colon (:) is constantly to be read, *in;* the parallels (=) are to be *understood* as read in the first *are* of each new divisor, and not afterward to be repeated. The phraseology of the table may be understood as from this example: that of 12s there are in 24, two; recited, 12s in 24, two.

Terms.

What is meant by terms in arithmetic? — Terms are the principal numbers entering into a question, and also the names given them.

What are the terms employed in division? — The terms used in division are, the divisor, the dividend, the quotient, and the remainder.

What is the first in order? — The divisor answers to a subtractor, and either specifies the number of parts to be valued, or the value of each part to be numbered by the division.

Does it not also receive some other name? — The divisor is also called the denominator, because it gives a proportional denomination to every single part, in value or number, into which a sum is to be divided; as a fifth, when the number of parts is five.

What do you say of the second term in division? — Dividend is a Latin word abbreviated; it answers to the minuend, and is the sum to be divided.

Has it any other name? — The dividend is also called the numerator, because it shows how many of the parts denominated by the divisor are to be taken.

What is the meaning of the third term? — Quotient is from a Latin word, signifying how many, so many; it answers the question, how often is the divisor contained in the dividend; or what is the value of the part denominated by the divisor.

What is the last term, and how represented? — The remainder is any portion of the dividend remaining after the divisor has been taken out every possible time; and therefore, because it gives only part of a unit to the quotient, is represented by a fraction, having the divisor for its denominator.

What are the factors under this rule? — The factors in division are the divisor and dividend; for the operation is effected by their means.

Demonstration of rule.

Valuation and severance of parts.

What is it to halve a thing? — To halve a thing is to divide its amount into two equal parts.

Then what will be the divisor? — To halve, we must divide by 2.

To third or quarter a number, how? — To third or quarter, we divide by 3 or 4.

To decimate? — To decimate a number, we divide by 10.

What is the meaning of to decimate? — To decimate is to take every 10th.

Then what is a decimal? — A decimal is one tenth ($\frac{1}{10}$).

Universally to make an equal valuation of certain known parts, what is the proceeding? — To assign to every known part an equal value we divide by the number of the parts themselves.

What results? — The result is the value of every single part.

Whence do you know it to be a true result? — We know the result to be true, for an apportionment of value is demanded, and value is given; and the correctness of the apportionment itself may be shown by its agreement with the table of divisions.

Does the work never exceed the table? — The work often exceeds the table; but many large divisors may be reduced to tabular factors, and every particular division may be shown to agree with the table.

How do you know the table itself to be right? — We know that the division and other arithmetical tables are right, when printed correctly, by their agreement with things.

But when I divide 1 by 4 ($\frac{1}{4}$), as in quartering a dollar, how does this agree with the table? — Fractions need no table to prove their correctness, if their quotients be but a repetition of dividend and divisor, as numerator and denominator.

Should decimal fractions however be divided like integers, how shall their correctness be proved? — Decimals, if divided like whole numbers, must show like figures, and be proved in like manner.

Is there no other mode of proving division? — Division is also proved by multiplication; for one is the reverse of the other.

Universally to make an equal severance into parts, how do you proceed? — The number of parts is equivalent to the turns of the value of each; by that value therefore we divide, when we would number the parts themselves.

To find any particular part? — To find any particular part, is to find its value or amount; therefore, as before observed, we divide by the number of the parts.

Integral division and fractional multiplication.

Divide five dollars into two equal parts; what is the result? —Five dollars, divided by 2, give two dollars and a half to each part.

Take half of 5 dollars, what shall we have? — Five dollars, taken to the extent of one half ($\frac{1}{2}$), produce 2 dollars and a half also.

In arithmetic, what means to take? — To take is to multiply.

In multiplication then of a number by one half, what is the procedure? — To multiply by $\frac{1}{2}$ we divide by 2.

What is the inference? — Division by whole numbers is equivalent to multiplication by fractions whose numerator is 1.

Integral and fractional parts of a quotient.

Divide 12 dollars into 4 equal portions, what will be the quotient? — Twelve dollars divided by 4, give 3 dollars to a part; because 3 can be subtracted four times from 12. $4 : 12 = 3$.

The quotient, 3 dollars, therefore is what? — Three dollars are one fourth part of 12 ($\frac{1}{4}$ of 12).

Integral or fractional? — Three dollars are a fractional part of 12, but an integral part of the quotient, because they are units represented by units.

Add a single dollar, what then will be the quotient? — A fourth part of 13 dollars, are $3\frac{1}{4}$ dollars; for the additional dollar gives an additional quarter to every fourth share. $4 : 13 = 3\frac{1}{4}$.

What part of the quotient will that quarter be? — The $\frac{1}{4}$ dollar will be the fractional part of the quotient, for it represents part of a unit.

Not of the 13? — It does not represent $\frac{1}{4}$ of the dividend, but one fourth of a unit, or single dollar; for if it represented a fourth of the dividend, it would make the quotient 6 dollars and more.

What is the inference, as to the value of fractions? — The fractional part of a quotient, and any fraction not a factor, is part of a unit.

Of what unit? — Of whatever unit is in question, money, goods, &c.

Why do you say, not a factor? — A fraction that is said to be of some other number, may be more or less than a unit; as $\frac{1}{4}$ of 12 is 3; $\frac{1}{2}$ of $\frac{1}{2}$ a dollar is $\frac{1}{4}$ of a dollar.

What are such fractions commonly called? — Fractions of other numbers are often called compound fractions.

Why do you call them factors? — They are factors, because they require something to be taken, and to take is to multiply.

Severance of units in the dividend.

What is the effect of division on the units of the dividend? — A divisor severs a part from every unit of the dividend, or may be considered as doing so.

How does this appear? — Since every unit goes to make up the whole, and the whole is divided, every unit also may, and sometimes must, be divided.

Can you show this by example? — In dividing 12 dollars into 4 parts, we may take a quarter from every dollar; for 12 quarters are equal to 3 dollars, the quotient.

How comes it then that 12 do not appear in the quotient? — As the dividend represents dollars, and the divisor will go in it an exact number of turns, the division is made by subtracting an entire share at once, or the amount of 12 quarters in 3 whole dollars.

Suppose an inexact number of turns? — If the dividend were 13 dollars, a unit must then actually be severed, a quarter would be subtracted with every share of 3 dollars and $\frac{1}{4}$ would be annexed to the quotient.

What would that fourth be called? — The fourth would be called the remainder; because it remains after every complete subtraction of the divisor.

How is a remainder figured? — Remainders are represented by fractions, for they give less than a unit.

How does this appear? — If the divisor would go once more in the dividend, another unit would be given to the quotient; when therefore it will no longer go once in the dividend, if any thing be given to the quotient, it must be a fraction.

Where to be placed? — To be placed on the right of the quotient.

Of what parts will the fraction be composed? — The remainder will form the numerator, and the divisor the denominator; for the divisor denominates the portion required, and the remainder shows how much is left toward contributing a unit to that portion.

Terms in division represented by common fractions.

What sort of a quotient would $\frac{12}{4}$ be? — Twelve quarters, *instead* of 3 dollars, would be a true quotient; for they are *of equal value.*

Of what would they be $\frac{12}{4}$? — They would be $\frac{12}{4}$ of a unit, or single dollar.

How does this appear? — Every unit of 12 dollars, severed by a divisor, can contribute, each, the fraction only of a unit to the part valued in the quotient; therefore the whole 12, appearing in the quotient, would number only so many fractions of a unit.

What then does every fraction represent? — Every fraction represents the terms of a division; for the denominator is a divisor, the numerator is a dividend, and the two, thus arranged, represent the quotient.

When the terms of a fraction are equal, what is the quotient carried out? — When numerator and denominator are equal, the quotient is a unit; for the divisor is then contained in the dividend once.

What may be inferred regarding the representation of whole numbers? — A whole number may be fractionally represented.

Arrangement of the factors.

In what order do the members of the divisions in the table stand? — In the divisions of the table, the divisor is on the left, the dividend on the nearest right, the quotient on the farthest right.

What is the symbol there placed after the divisor? — After the divisor, and preceding the dividend, a colon (:) is placed.

What is the quotient equal to? — The quotient is equal to the turns and part turns of the divisor in the dividend.

Then how might you separate the quotient from the other members? — The quotient may be separated from divisor and dividend by the sign of equality (=).

Manner of dividing.

How many 12s are there in 120? — Twelves in 120 are 10.

$$12 : 120 = 10.$$

Is it possible to make two steps of that division? — I might say, 12s in 12, one; in nought, none.

At what places did you then begin? — I began with dividing the highest places only of the dividend.

How would you thus divide 121? — In dividing 121 I might say, 12s in 121, ten, and 1 over; for ten 12s are 120.

What would you do with the unit over? — The remaining unit I might annex to the quotient, 10, as a fraction, having 12 for its denominator. $12 : 121 = 10\frac{1}{12}$.

Make the dividend 1214, how will you proceed? — To divide 1214 by 12, I might say, 12s in 12, one; 12s in 1,

none, or no entire 12; 12s in 14, one, and 2 over. 12 : 1214 $= 101\frac{2}{12}$.

Why do you not add 1 to the 4 on the right, and say, 12s in 5? — The 1 is in the place of tens, and must not be carried as a unit.

Why do you set down a cipher for the place in the dividend which does not contain the divisor once? — Every place of the dividend not containing the divisor once must be noted by a cipher in the quotient, to show the true place of figures on the left; for the divisor may be taken in thousands, or hundreds, or tens; and the notation of the quotient must be such as to indicate which.

Since the left hand place of the dividend does not contain the divisor, why do you leave it without note? — Ciphers are never placed on the left of whole numbers, for they could not denote place beyond the highest.

What are the highest places of the last dividend equal to? — The two highest places of 1214 are equivalent to 1200.

How then is the true quotient found by your manner of division? — When I say 12s in 12 of that dividend, one, I mean one hundred; and this appears in the quotient, by the noting of every place of the dividend on the right of the significants in 1200.

Divide 1214 by 121; how will you proceed? — The dividend 1214 contains the divisor, 121, ten times; for the places of the dividend are but four, and the three highest places have the same digits with the divisor; I will therefore set down 1 for the highest place in the quotient.

What is the value of those three places? — The three highest places of the dividend are in value, 1210.

Is the divisor contained in the remaining place? — The divisor being no longer contained in the dividend, I will add a cipher on the right of the former quotient, and annex the remaining 4 as a fraction. 121 : 1214 $= 10\frac{4}{121}$.

Divide 1214 by 109; how will you proceed? — The divisor 109 is contained in the three higher places of the dividend once, for which I set down 1 in the quotient; and subtracting 109 from 121, prefix the 12 over to the remaining figure of the dividend; which becomes 124, containing the divisor once also, and leaving a remainder of 15, which I annex to the quotient.

109 : 1214 $= 11\frac{15}{109}$
109
——
124
109
——
15

Let the divisor be 76; how will you proceed? — Seventy-six are contained once in 121; the difference is 45, making, with the 4 units of the dividend, 454.

How will you find the turns of 76 in 454? — Seven are contained in 45 six times, with something over; but six 6s are 36; the units therefore of the product will be more than those of the dividend, while the tens and hundreds are equal; and a greater number cannot be subtracted from a less.

$$
\begin{array}{r}
76 : 1214 = 15\frac{74}{76} \\
76 \phantom{= 15\frac{74}{76}} \\
\hline
454 \phantom{= 15\frac{74}{76}} \\
380 \phantom{= 15\frac{74}{76}} \\
\hline
74 \phantom{= 15\frac{74}{76}}
\end{array}
$$

Can you not try a smaller quotient? — Five times 76 are 380, leaving a difference of 74, to be annexed fractionally to the quotient.

Divide 45 by 9; what is the quotient? — Nines in 45 are 5.

$$9 : 45 = 5.$$

Of what are 45 the product? — Forty-five are the product of 9 and 5.

Then what is a dividend? — A dividend is a product.

How produced? — Its factors are the divisor and quotient.

What proof does this afford of division? — *If the product of divisor and quotient be equal to the dividend, the division is right.*

What will be the places of a product compared with those of its factors? — An entire factor of larger amount than the same number of digits in the highest places of the product gives to the product one place more than itself (the factor) contains.

Can you assign the reason? — If it did not, the digits of the factor being of greater value than an equal number in the highest places of its product, multiplication would be diminution.

Can you exemplify this? — Nine and five are factors of 45; the highest place of the product has a digit of less value than either factor, but the entire product contains a place more than either; so of 108, the factors of which are 9 and 12.

Can you now give me a maxim derived from these considerations? — When a divisor is not once found in an equal number of the highest places of the dividend, its product by the quotient will have one place more than itself (the divisor) contains.

Why may it not have two places? — Multiplied into a single figure, it cannot have more than one place additional; for a single ten could give it no more.

Ciphers on the right of divisor.

What effect have ciphers on the right of a divisor?—Ciphers on the right of a divisor cannot affect the integral part of the quotient, so far as it extends; since they cannot, by subtraction, change the figures of the dividend.

In what manner then will they affect the quotient?—They diminish the number of places in the quotient; for a divisor ten times greater must give a quotient ten times smaller; and every cipher makes the divisor ten times greater than if there were none.

Suppose the ciphers were cut off, and neglected?—If ciphers on the right of a divisor be cut off, and as many places cut off on the right of the dividend, the quotient will be unchanged.

How does this appear?—Ciphers on the right of a divisor diminish the places of the quotient, one for every cipher; a dividend also reduced ten, a hundred, a thousand, times, and so forth, will reduce the quotient in the same degree; therefore if a number of places equal to ciphers on the right of a divisor, be cut off from the right of the dividend, the ciphers may be neglected.

Neglected through the entire operation?—In annexing a remainder, the ciphers must be made part of the denominator; for the remainder is a part of the dividend from which no places are cut off.

What will that remainder consist of?—The remainder will consist of any number left on the last subtraction, prefixed to whatever places were cut off from the dividend; for all the places of the dividend, without exception, are to be divided, integrally or fractionally.

Can you exemplify this subject?—Let the number 2760 be divided by 500; if the ciphers be cut off from the divisor, and the two right hand places from the dividend, we may then say, 5s in 27, five, and 2 over; these, prefixed to the places severed from the dividend, make 260, to be annexed fractionally to the quotient, having the whole divisor for its denominator.

$$5{,}00 : 27{,}60 = 5\tfrac{260}{500}.$$

$$\begin{array}{r} 500 : 2760 = 5\tfrac{260}{500}. \\ 2500 \\ \hline 260 \\ \hline \end{array}$$

What is it to divide by 10?—Division by 10, or by a unit having ciphers on the right, is the severance of a place from the dividend for every such cipher in the divisor; with the annexation of any digits of value in places cut off, as a remainder.

$$100 : 2760 = 27\tfrac{60}{100}.$$

Why are places thus severed on such a division?—Because

division is the reverse of multiplication; to multiply by 10 and its powers, is to annex a place for every cipher after a unit in the multiplier; therefore division by 10 and its powers is the severance of a place or places.

What kind of fractions do the remainders become? — The digits severed, if they are not ciphers merely, and lost, become decimals, though figured in the manner of common fractions.

Can you think of no independent reason for this mode of division? — Numbers advance toward the left in the proportion of one place for every tenfold increase of value; therefore they lose a place for every diminution to a tenth of their former value.

Division from the left.

In subtraction we begin on the right; can you tell why? — We begin to subtract on the right, because we subtract by single figures; figures of every amount occur, as well in lower places as in higher; so that if the operation were begun on the left, few subtractions could be made without altering differences already obtained; for we are obliged often to take from higher places, that we may be able to subtract from lower.

In division we begin where? — Division is begun on the left, because distribution naturally descends from higher to lower values, and because the inconvenience of subtracting first from higher places is obviated.

In what manner? — The divisor, or its multiples, is at every turn subtracted from an equal or greater number; and all excess being prefixed to remaining places of the dividend, the whole is brought to subtraction, without the necessity of preceding figures.

Division by submultiples.

When may submultiples be employed in division? — Submultiples may be used in division, whenever the divisor is a multiple having factors within the table.

To what advantage used? — In using tabular numbers, our memory serves us; the necessity of writing products in division is thus obviated.

What is the manner of using them? — With submultiple divisors, a former quotient is made a succeeding dividend.

How do you show that the quotients of multiple and submultiples agree? — The last quotient by submultiples is equal to the quotient that would be given by their multiple; because the last submultiple divisor is as much less than the multiple, as the last dividend, or preceding quotient, is less than the original dividend.

Can you exemplify this? — If a dividend be 500, and its divisor, 45, the submultiples are 5 and 9; dividing first by 5, the last submultiple is but a 5th part of the multiple divisor, and the last dividend is but a 5th part of the original dividend.

$5 : 500$

$9 : 100 = 11\frac{1}{9}$.

$45 : 500 = 11\frac{5}{45}$.

$$\begin{array}{r} 45 \\ \hline 50 \\ 45 \\ \hline 5 \end{array} \qquad \frac{5}{45} = \frac{1}{9}.$$

Fractions in proof.

In proving division, what must be done with the fractions of the quotient? — In proving division the entire quotient must be multiplied into the divisor; the fractions therefore as part of the quotient.

What is the denominator of those fractions? — The denominator of such fractions is the divisor itself.

What is the office of a denominator? — A denominator is a divisor.

Make the divisor then a multiplier, what is the result? — The consequence of making a denominator a multiplier is to leave the numerator a whole number; for the fraction is then equally multiplied and divided.

Can you exemplify this? — If $\frac{1}{2}$ a dollar be twice taken, that is, be multiplied by 2, the product is the numerator, or 1 dollar.

What will you do with whole numbers thus obtained? — In the proof of division the numerators of fractions must be added to the product of divisor and integral part of the quotient; because they form part of the dividend.

Can you now recite the rule and the proof?

RULE OF DIVISION.

To divide, set down the divisor, followed by a colon distinct; set the dividend on the right, separated, by the mark of equality, from the quotient. Consider how often the divisor is contained in the highest place or places of the dividend; and, if necessary, mark the number of times aside; multiply it into the divisor, and set the product under the dividend, beginning, if the divisor will not go once in as many of the higher places of the dividend, at one place farther to the right than may be contained in the divisor. If the product exceed the dividend taken, multiply the divisor by a unit less than before; if it be equal or less, make the factor found the high-

est place in the quotient; and if less, subtract; if the difference be less than the divisor, prove the subtraction, and to the difference annex the left hand undivided figure of the dividend; if the difference be none, from the product being equal, bring down the succeeding left hand undivided figure; then, if the new dividend be equal to or greater than the divisor, proceed as before; if less than the divisor, annex also the succeeding undivided figure, should any remain undivided; to the quotient annexing a cipher for every figure more than one thus taken to form a new dividend; and also for the last figure of the dividend, if insufficient to form a new dividend. After this manner multiply, subtract, annex, or bring down, till the dividend will no longer give a unit or cipher to the quotient. Set any final remainder in the quotient as a numerator, and the divisor for its denominator.

Should there be ciphers on the right of the divisor, cut them off by a comma, and cut off as many places on the right of the dividend; annex the places so cut off to any other final remainder; if none other, make then the final remainder, and the whole divisor, in either case, its denominator.

Should the divisor be tabular, or of easy operation, its products may be omitted, the figures of the quotient written as they are found, intermediate remainders mentally prefixed to succeeding lower places, a cipher noted for every place more than one divided at a time, and the final remainder fractionally noted.

Should the divisor be a multiple, having factors within the table, divide by submultiples successively, making a preceding quotient the succeeding dividend.

Proof. Multiply divisor and quotient, including any fraction, into each other; if the product be equal to the dividend, the division is right.

APPLICATION.

After the ample explanation, and the numerous examples, already given under this rule, little more can be needed for its illustration. It is however advantageous to concentrate examples of the different cases; and the manner of the proof may yet be imperfectly understood; in submultiple division it must so continue for some time.

1. Tabular division.

$$
\begin{array}{rl}
11 : 67300213 = & 6118201\tfrac{2}{11} \text{ ans.} \\
 & 11 \\
\hline
 & 67300213 \text{ proof.}
\end{array}
$$

2. Long division with ciphers cut off.

906,00 : 563109,08 = $621\frac{48308}{90600}$ answer.

```
   5436             90600
   ----          621 48308/90600
    1950         -----
    1812            48308
    ----            90600
     1389          181200
      906          543600
      ---          --------
      483          56310908 proof.
```

3. Submultiple division.

Divisor, $54 = 6 \times 9$

```
9 : 815916                   15109 30/54
    ------                      54
6 :  90657 3/9                -----
     -------                  60466
     15109 30/54 answer.     75545
                             ------
                             815916 proof.
```

In proving the 1st example, 2, the numerator of the fraction, are added to 11, the first partial product. In the proof of the 2d example, the numerator, being too large to be added to partial products, is made the first separate product, as directed by the rule of multiplication; and the fractional factor is made the multiplier, that the fraction may more conveniently be on the right of every other digit. When a quotient, for convenience' sake, is placed under the dividend, as in example 3, its highest place is set under the lowest of the dividend which first contains it. The division of the fraction, $\frac{3}{9}$, in the same example, cannot be fully understood, till the subject of fractions has been studied; it may be observed, however, that the number 3, remaining on the division of 57 by 6, is multiplied into 9, the denominator of the fraction, and their product added to the numerator, making 30; and that the denominator itself is multiplied into the divisor, 6, and their product made a new denominator. Hence the following occasional rule, which may be of use till a knowledge of fractions shall supersede it. *Multiply any new remainder into the denominator of the remainder last preceding, adding their product to the numerator; multiply the denominator also of a former remainder into the actual divisor, for a new denominator.* The learner is now recommended to solve the examples already given, copying the factors on his slate, and reciting the results, or comparing them with our exemplifications; after which he will proceed to the farther consideration of multiples, when it will be necessary to repeat a few of our former questions.

Multiples.

What are multiples? — **Multiples are the product of two or *more numbers.***

What are submultiples? — Submultiples are the numbers producing a multiple.

What submultiples are of use in arithmetic? — The submultiples of use are chiefly those which fall within the table.

In what operations are they employed? — They are employed in multiplication and division.

What is the advantage derived from them? — In multiplication they save the adding of products; in division, the writing of them.

How are they found? — Submultiples are found by trying tabular factors; as of 27, by considering that nine 3s are 27.

What is an even number? — An even number is one that may be halved without a fraction, and is therefore divisible by 2.

What is an odd number? — An odd number is one that cannot be halved without a fraction; as 7.

A number divisible more than once by 2, is divisible by what else? — A number that remains even after division by 2, is divisible by 4; for remaining even it is again divisible by 2; by the first division the number is halved, by the second it is quartered; therefore it is divisible by 4.

What is the first power of 2? — The first power of 2, or of any number, is that number itself.

What distinctive meaning has the term power? — Power has a reference to the products of a number multiplied into itself, produced therefore without the aid of any other number.

What are the second and third powers of 2? — The second power of 2 is 4, the third is 8; for twice 2 are 4, and twice 4 are 8.

Have any of these powers peculiar names? — The second power of a number is called its square; the third power, its cube.

Numbers ending in a cipher are divisible by what? — Every number having a cipher on the right is divisible by 10; for to divide by 10, is only to cut off a place.

By what other number is it divisible? — Every number divisible by 10 is divisible also by 5; because any number of tens is exactly double the number of fives.

Of what numbers are 10 a multiple? — Ten are a multiple of 5.

Numbers having 5 in the units' place are divisible by what? — Numbers ending in 5 are divisible by 5; for the place precedent is of tens, which are divisible by 5, and 5 are measureable by 5.

Universally, what numbers are divisible? — All numbers ending *in 2, 4, 5, 6,* 8, or 0, are divisible; for the 5 is divisi*ble by 5, the rest* may be halved, &c.

Is every number a multiple? — All numbers are not multiples; for they are not the perfect product of any two or more numbers; as 17.

Can you now state some maxims on this subject?

Maxims to find submultiples.

All even numbers are divisible by 2. All numbers having a cipher on the right are divisible by 10 and by 5. All numbers ending in 5 are divisible by 5. All numbers divisible by submultiples are divisible by their multiples.

Since division is proved by multiplication, how is the latter proved? — Multiplication is proved by division; for division subtracts what multiplication adds.

How is this more particularly shown? — The product in multiplication is the repetition of one of the factors by the other, as many times, and to the extent of as many parts, as there are units and parts of a unit in that other.

What is the inference? — Therefore, if the multiplication be right, and one factor, by division, be subtracted from the product as many times and parts as it is contained in the product, its quotient will be the other factor.

Can there be any remainder in this case? — In the proof of multiplication by division there can be no remainder, unless there be a fraction in the factor sought; for if there should be any other remainder, either the division or the multiplication must be wrong.

What would you do in that case? — I would then re-examine the work throughout, endeavouring to detect the source of the error and to correct it.

Can you now formally state the manner of proof?

PROOF OF MULTIPLICATION.

Divide the entire product by one of the factors; if the quotient be the other factor, the multiplication is right.

The proofs of division, in the examples lately given, serve equally to show the manner of proving multiplication. If deemed advantageous, the examples already given in multiplication may be wrought anew, and proved by division.

Examples to be wrought and recited.

How much is

1. $\frac{1}{2}$ of 3976010? $\frac{1}{3}$ of 567901291? $\frac{1}{4}$ of 7103561009? $\frac{1}{5}$ of 1534100821?
2. $\frac{1}{6}$ of 6128342? $\frac{1}{7}$ of 371869057? $\frac{1}{8}$ of 7903799766? $\frac{1}{9}$ of 3124658070?
3. $\frac{1}{10}$ of 5055954? $\frac{1}{11}$ of 715006770? $\frac{1}{12}$ of 5046897544? $\frac{1}{13}$ of 9637051009?

What are the contents of

4.	9 : 6019908 ?	11 : 207934007 ?	12 : 180701321 ?	13 : 7310602189 ?
5.	9 : 5450119 ?	11 : 308944751 ?	12 : 904623769 ?	13 : 6790032191 ?
6.	90 : 3040350 ?	110 : 620105674 ?	120 : 987613402 ?	130 : 8305683276 ?

What is the value of

7.	40026891÷17 ?	14768007096÷19 ?	7302856735÷23 ?	8064566050÷29 ?
8.	29815671÷37 ?	7690583114÷41 ?	3874400668÷58 ?	9486150433÷62 ?
9.	11994605÷74 ?	9143704362÷82 ?	2090077359÷91 ?	7890123716÷98 ?

How many turns of

10.	133 : 2070618 ?	179 : 557819514 ?	238 : 178564001 ?	394 : 603805916 ?
11.	4830 : 5463971 ?	5160 : 330050690 ?	62900 : 4007190372 ?	75800 : 858140095 ?
12.	8641 : 1699032 ?	9976 : 913278670 ?	102510 : 827362871 ?	2684 : 5980506127 ?
13.	73900 : 689791 ?	8045 : 760159070 ?	9694 : 60141529168 ?	93200 : 904671562 ?

How many times are

14.	14 : 301246118 ?	15 : 7110206145 ?	16 : 40591820012 ?	18 : 97222396088 ?
15.	21 : 798107269 ?	36 : 1505678098 ?	42 : 620305817966 ?	55 : 765400108300 ?
16.	88 : 31977183 ?	96 : 6013021897 ?	144 : 111671923450 ?	169 : 25800397069 ?
17.	630 : 545676561 ?	720 : 2903377104 ?	990 : 706556071842 ?	4800 : 71096056710 ?

The manner of reciting work in division has already been given in the course of our demonstrations; every term of the question is first to be enumerated and named.

ROMAN NUMERALS.

What manner of representing numbers has at times prevailed in Europe, beside that of figures? — The Romans represented numbers by characters, or marks, resembling letters of their alphabet.

What are such representations called? — Letters thus representative of numbers are called Roman numerals.

How are the three first numbers represented in this method? — One, two, three, are represented by as many upright strokes, now printed like the capital letter I. I·II·III

The next number how? — Four are represented, sometimes by four upright strokes, at others by an upright before a V. IIII or IV

From five to eight in what manner? — Five are represented by the letter V; six, seven, and eight, by the same, with one, two, or three, strokes following it. V· VI· VII· VIII

The two next, how represented? — Nine are represented by an upright before an X; ten by X alone. IX· X

From eleven to nineteen, how? — From eleven to nineteen, the numerals expressive of an excess above ten are annexed to the right of the character for ten.

XI· XII· XIII· XIV· XV· XVI· XVII· XVIII· XIX

How are the succeeding tens represented? — Twenty and thirty are represented, the first by two, the latter by three, characters of ten.

XX· XXX

The intermediate numbers how? — From ten to a hundred, any excess above an exact number of tens is annexed to the right of the tens.

XXI· XXXIV, &c.

How are forty and fifty represented? — Forty are represented by an X preceding an L; fifty by L alone.

XL· L

To eighty, in what manner? — Sixty, seventy, and eighty by an L, with one, two, or three, characters of ten appended on the right.

LX. LXX· LXXX

The next ten how? — Ninety are represented by an X before a C.

XC

Hundreds, how? — Hundreds are represented by as many characters of C, to four hundred; five hundred being represented by D.

C· CC· CCC· CCCC

D

Ten hundred, how? — A thousand is represented by the letter M.

M

Were these characters entirely arbitrary? — Some of them appear to have a meaning, C being the first letter of *centum*, the Latin for a hundred; M, the first letter of *mille*, the Latin for a thousand; while ten are represented by two fives inverted to each other.

Has any one of the characters a more natural origin? — An upright stroke, being as simple a mark as nature furnishes, and answering to a finger, seems well fitted to represent a unit, and has perhaps been universally adopted.

Does the arrangement you have been describing exhibit any marks of system? — It may be perceived, on the examination of Roman numerals, that a less number before a greater is subtracted; but *after* a greater is added; as one before a five represents four; after a five, it represents six.

For what purpose are these numerals now chiefly employed? — Roman numerals are now chiefly employed for the dates of inscriptions and publications; to distinguish also between the chapters and smaller divisions of books.

Are these characters constantly represented in the same position? — We sometimes see the characters reversed, or inverted; in which case they take a different value.

What are the instances? — Instead of a D, an inverted, or reversed, C after an I, is made to represent five hundred; a second, and every additional reversed C on the right, serves as a multiplication by one or more tens; and this is possibly the origin of our mark of place only.

IↃ five hundred
IↃↃ five thousand
IↃↃↃ fifty thousand

Is there no other instance? — Instead of M, a thousand is sometimes denoted by C before I, followed by a reversed C.

CIↃ one thousand

What other sign of multiplication had the Romans? — A short horizontal line placed over a numeral is equivalent to multiplication by one thousand.

Can you give me an example? — V with a line over it denotes five thousand; X, with a similar line denotes ten thousand.

$\overline{\text{V}}$ five thousand
$\overline{\text{X}}$ ten thousand

COMPUTATION.

The following examples are introduced as involving only the first part of arithmetic, or that of integers. The exercises for the learner, with the exception of a few fractions and a few (borrowed from other authors) in exchange, will be without results stated; as he is expected strictly to prove every step of a process, and our rules are illustrated and exemplified in the most ample manner. In very many instances the answers supplied by arithmetical treatises supersede all exercise of the understanding in statement, and nearly so in operation.

Examples to be wrought, proved, and recited.

1. Arithmetic is said to have been brought from Egypt to Greece by the philosopher Thales, in the year 600 B· C· how many years have elapsed since?

2. Notation by the 9 digits and zero known to the Hindoos, A· D· 600; how long ago was that?

3. If a man's age be 35 years more than his son's, who is in his 15th year, how old is the father?

4. How many days are there in the first 11 calendar months?

5. How many days are there from 12th January of one year, to 12th March of the following year?

6. What would be your age, were you thrice as old as you are ?

7. How many strokes on the clock bell are striken in 24 hours ?

8. What is the age of a man who was 43 years old 29 years ago ?

9. Newton was born, A· D· 1642; he died in the 27th year of the following century ; at what age ?

10. The number of Jews in Europe has been calculated to be one million nine hundred eighteen thousand fifty-three ; in Asia, seven hundred thirty-eight thousand ; in Africa, five hundred four thousand ; in America, five thousand seven hundred ; in Australia, fifty ; what is their probable number throughout the world ?

11. The eastern coast of America was discovered by Ojéde and Amerigo Vespucci, (if not previously by Cabot,) A· D· 1499 ; how many years ago ?

12. The deluge happened about A· M· 1656 ; the birth of Christ about A· M· 4000 ; how long did the deluge precede the Christian era, and what length of time has elapsed since the deluge ?

13. The difference between two numbers is five hundred sixty-nine thousand five hundred ten ; the less is one hundred seventy-six thousand three hundred ninety-seven ; what is the greater number ?

14. The difference between two numbers is three hundred forty-eight thousand one hundred two ; the greater is one million five hundred thousand ; what is the less ?

15. How old is a man that was born in the 89th year of the last century ?

16. What number taken from six thousand four will leave five hundred eight ?

17. What number increased by eight thousand and nineteen will amount to ten thousand ?

18. Twenty thousand dollars are bequeathed in three shares, one share of which is equal to the other two ; what will be the amount of each ?

19. Borrowed, a fifty dollar bill, a hundred dollar bill, and a twenty dollar bill ; since paid, five three dollar bills, a twenty dollar bill, six ten dollar bills, and seven one dollar bills ; what remains to be paid ?

20. If a man travel during 6 days at the rate of 40 miles a day, and another, on the same road, and during the same time, travel only 23 miles a day, how far will they be apart on the evening of the 6th day ?

21. Sold a piece of land for $855, which cost $618 ; what was the gain ?

22. An army consisted of 49 battalions of 700 men each; what was the entire force?

23. If a man lay up six cents a day, how many cents will he have saved by the year's end?

24. Travelling at the rate of 5 miles an hour, during 7 hours of the day, what distance shall I have made in 5 days?

25. A ship averaging 9 miles an hour will have sailed how far in 7 days?

26. Having a thousand miles to go in 30 days, how many miles a day must I travel, resting on the sabbath?

27. A nursery contains 19 rows of 37 trees each, and three rows of 19 each; how many trees are there in all?

28. Suppose a town to contain 235 houses, averaging 2 families in each, and every family 5 persons; what is the population?

29. If the inhabitants of a town be three thousand one hundred seventeen, and the number of houses six hundred fifty-nine, what will be the nearest possible average to a house?

30. Of what number are 17, 31, 49, and 57, factors?

31. If the population of France, fifteen years ago, were thirty-one millions, and its average annual increase be six thousand five hundred thirty-six for every million, what is its present population?

32. Forty-five hundred dollars, distributed among a number of persons, gave thirty-six dollars to each; how many persons shared?

33. The dividend is fifteen million four hundred three thousand four hundred twenty, the quotient is nine thousand six hundred sixty-four, with a remainder; what is the divisor?

34. Twenty-nine chests of tea averaged 112 pounds weight each; the chests alone, 17 pounds each; what was the weight of the tea alone?

35. If the motion of the earth in its orbit be taken at a million and a half of miles in a day, and no computation of the return of a comet have hitherto been verified within nineteen days, how far distant might the two bodies be from each other, though calculated to come into collision on a day specified?

36. The oldest treatise on arithmetic known is by Euclid, forming the 7th, 8th, and 9th books of his Elements, of about the date, A. C. 300; what time has elapsed since?

37. The first printed English work on arithmetic was by Tonstall, bishop of Durham, A. D. 1522; how many years have since elapsed; and what was the interval between the works of Euclid and of Tonstall?

After these examples have all been wrought on the slate and recited, many of them may, with great advantage, be solved anew mentally, on interrogation.

We now proceed to fractions, the most intricate of the elementary parts of arithmetic, intricacies however which are easily unravelled, by any one who means not to throw away the hours ordinarily devoted to arithmetic in a course of education. But whether of easy or of difficult acquisition, no man must pretend to be an arithmetician, who has not made the acquisition his own: it is of absolute necessity to any further progress; nor have we advanced even thus far, without perceiving, that the consideration of fractions enters often into very simple calculations.

ARITHMETIC.

PART II.

COMMON FRACTIONS.

Definitions. Notation.

With what may fractions be compared? — Fractions are parts of a unit, as fragments are of a thing.

Of a unit in the strictest sense? — Fractions are of a unit in the strictest sense, when they stand alone; and when used of a number larger than unity, as of a hundred, that number may be taken collectively as one whole.

How many kinds of fractions are there? — Fractions are common and decimal.

How are they distinguished? — Common fractions are represented by two numbers, called numerator and denominator; placed, the former above, the latter below, a short line.

What is the meaning of those terms? — The denominator is so called, because it gives a proportional denomination to every part of a whole; as a 5th; the numerator, because it numbers the parts taken out of the whole; as four fifths ($\frac{4}{5}$).

How are the parts of a dollar represented? — Dimes, cents, and mills, are distinguished from dollars by a point on the left.

What kind of fractions are they? — Dimes, cents, and mills are decimal fractions of a dollar.

What is $\frac{1}{2}$ a dollar? — Half a dollar is a common fraction of a dollar, equal to $.50.

Is there any essential difference then between the two kinds of fraction? — Common and decimal fractions differ from each other in notation only.

Reduction.

Do you recollect the sign of addition?

Employing figures and fractions, how might you add together two half dollars? — Two half dollars must be added together by their numerators; for only in this way can we make a dollar of them. $\frac{1}{2}+\frac{1}{2}=\frac{2}{2}=1.$

How does it appear that the sum is a dollar? — The numerator and denominator being equal, and the denominator being a divisor, the quotient must be a unit.

What is the inference as to a possible representation of units? — *A unit may always be represented in a fractional form, by making numerator and denominator equal.*

Would it be a misrepresentation of value, to add another half dollar in the fractional form? — A third half dollar may be added to the preceding two halves, for numbers represent things under whatever form things themselves may assume.

$$\tfrac{1}{2} + \tfrac{2}{2} = \tfrac{3}{2} = 1\tfrac{1}{2}.$$

What kind of fractions are thus formed? — Two halves and three halves are not properly called fractions, for the former are equal to a unit, the latter to a unit and a half.

What then would you call them? — *Fractions whose numerators equal or exceed their denominators are improper fractions.*

What would be a suitable name for the unit and a half? — *Numbers made up of integers and fractions may be called mixed numbers.*

Can we add another half dollar to the fraction? — Four in the numerator and two in the denominator will truly represent four half dollars. $\tfrac{1}{2} + \tfrac{3}{2} = \tfrac{4}{2} = 2.$

The numerator is a dividend; of what is it in this case the product? — Four, the numerator of the improper fraction, four halves, is the product of the denominator multiplied into the whole number 2, the equivalent of the fraction.

What is the inference, as to the representation of whole numbers? — *Any whole number may be represented in a fractional form, by multiplying it into the proposed denominator, and making their product the numerator.* Why? — Because the division, carried out, must reproduce the whole number in the quotient.

Change of terms without alteration of value.

How much are $\tfrac{1}{2}$ and $\tfrac{1}{4}$ added together? — We know that one half and a quarter, added together, make three quarters. $\tfrac{1}{2} + \tfrac{1}{4} = \tfrac{3}{4}.$

These fractions however have different denominators, and are different parts; by what means are they brought together in one sum? — Half is equal to two quarters; these, added to one quarter, make three quarters. $\tfrac{2}{4} + \tfrac{1}{4} = \tfrac{3}{4}.$

How is this change brought about arithmetically? — By multiplying numerator and denominator of $\tfrac{1}{2}$ into 2, the fraction is made to express quarters.

Is the value in this and every similar case unchanged? — A

whole number is changed to an improper fraction, by giving it a denominator, and multiplying it into the same for a numerator; therefore a fraction whose denominator and numerator are multiplied into the same number, remains unchanged in value.

What then is the change made? — The change, by multiplying both terms of a fraction, is of larger into smaller parts, for the greater the divisor, the smaller are the parts; the number however is proportionably increased.

What is the general inference? — *To multiply numerator and denominator into the same number, leaves a fraction unaltered in value.*

What is the effect of diminishing a divisor? — To diminish a divisor is to increase the quotient; for the smaller the divisor, the larger are the parts.

What is the effect of increasing a dividend? — To increase a dividend is also to enlarge the quotient.

Then if, instead of increasing a numerator, we should diminish its denominator in the same proportion, would the quotient be the same? — *To divide a denominator, or to multiply a numerator by the same factor, gives the same result.*

Since to multiply numerator and denominator into the same number, makes no change in the value of the fraction, what will be the effect of a reverse operation? — *To divide numerator and denominator by the same factor, leaves a fraction unaltered in value.*

Wherefore? — Because the quotient will be the same; for however much the dividend is reduced, the divisor is proportionably reduced.

Can you exemplify this? — If the terms of $\frac{2}{4}$ be each divided by 2, the quotient is $\frac{1}{2}$; an equal value.

What is the change here made? — The change made is in the dimensions, consequently in the number, of the parts.

What are the circumstances then affecting the value of fractions? — Largeness of number does not at all affect the value of fractions, but the comparative difference only of numerator and denominator.

Can you exemplify this? — Half a dollar is of equal value with fifty cents, or hundredth parts of a dollar; for 2 are the double of 1, and 100 are the double of 50. $\frac{1}{2} = \frac{50}{100} = .50$.

Change of terms in addition and subtraction.

In what manner did you lately add $\frac{1}{4}$ to $\frac{1}{2}$? — By multiplication of the numerator and denominator of $\frac{1}{2}$ into 2, it became $\frac{2}{4}$; and by adding the numerators, the sum total was $\frac{3}{4}$.

Is it possible to add them together in any other manner? — Fractions of different denominators cannot be added together, without confounding parts of different dimensions and values.

Hence what may be inferred as to the addition of fractions? — *Fractions are added by their numerators, their denominators being the same, originally or by reduction.*

How would you subtract $\frac{1}{4}$ from $\frac{1}{2}$? — To subtract $\frac{1}{4}$ from $\frac{1}{2}$, I should proceed in the same manner, reducing $\frac{1}{2}$ to $\frac{2}{4}$, and subtracting the numerator, 1, from the numerator, 2.

What is the inference as to the subtraction of fractions? — *Fractions are subtracted by their numerators, their denominators being the same, originally or by reduction.*

Whence the necessity? — Fractions of different denominators cannot be subtracted, one from another, without change to the same denominator; or, to instance it, the subtraction of a quarter would take away half.

What is the name given to these changes? — Equal changes of both terms of a fraction are called reduction.

What is essential to them? — *Numerator and denominator must both be equally changed in every reduction of fractions.*

Why? — Lest their proportions should be changed.

What is the purpose of the rule? — Reduction is used to bring different fractions to similar denominators, and single fractions to smaller denominators.

Why to smaller denominators? — Because small numbers are more easily computed than large numbers.

What example have we had of this latter reduction? — The reduction of $\frac{2}{4}$ to $\frac{1}{2}$ is of this kind.

Can you now define the operation? — *Reduction of fractions is their change to a smaller, or to a common, denominator, without change of their value.*

Common measures.

What is the signification of measure? — Measure is the estimate of magnitude.

May we apply the word measure to number? — Measure may be spoken of number, for magnitude is represented by number; as in feet and inches.

How is measure applied to magnitude? — Measure divides magnitude into known and specified parts, called dimensions.

Then what will be the measure of a number? — *The measure of a number is any factor or power which divides it without remainder.*

Why without remainder? — That is not a perfect measure which does not measure the whole.

What is a common measure? — *A common measure is any factor or power common to two or more numbers, which will therefore divide them without remainder.*

What is a prime number? — A prime number is one that can be measured by a unit only.

Is there any other general description of such numbers? — All numbers are prime that are not multiples.

Reduction to least terms.

What is meant by the least terms of a fraction? — The least terms of a fraction are the smallest numerator and denominator in which it can be expressed.

How were $\frac{2}{4}$ reduced to $\frac{1}{2}$? — Two fourths were reduced to one half by dividing numerator and denominator by 2, a measure common to both.

Can any farther reduction be made? — One half can be reduced to no smaller denominator, for we can no longer divide the numerator.

What may this reduction be styled? — The fraction is now, by division, reduced to its least terms.

Can no greater common measure of $\frac{2}{4}$ be found? — The greatest common measure of $\frac{2}{4}$ is 2.

Hence what rule may be derived? — *To reduce a fraction to its least terms, divide numerator and denominator by the greatest common measure.*

To common denominators.

One half is added to one fourth by what means? — The $\frac{1}{2}$ and a fourth ($\frac{1}{4}$) are added together, by giving those fractions the same denominator, and adding their numerators.

How were those fractions so reduced? — A common denominator was obtained by increasing the less as many times as it is smaller than the greater; 2, the least denominator, being twice as small as 4, the greater, we double the least, to make them equal. $\frac{1 \times 2}{2 \times 2} + \frac{1}{4} = \frac{3}{4}$.

Arithmetically, how do we know what proportion one number has to another? — We learn what part one number is of another by division.

Does not this enable us to find a common denominator? — *Fractions of denominators the submultiples of larger denominators are reduced to the same larger denominators, by multiplication of both terms into the corresponding submultiples.*

How shall we add $\frac{1}{3}$ to $\frac{1}{2}$? — Two not being a submultiple

of five, no way of bringing the fractions to a common denominator presents itself, but that of multiplying the denominators together.

How will you compensate for this, as to the numerators? — If the numerator also of the one be multiplied into the denominator of the other, the compensation will be exact; for thus both terms are increased alike.

Is not this another mode of finding a common denominator? — *Fractions of different may be reduced to common denominators, by multiplying all the denominators, one into another; and every numerator into every denominator not its own.*

Of improper fractions to other forms.

How may a whole number, fractionally represented, be restored to its proper form? — The denominator of a whole number is a measure of the numerator; therefore, by carrying out the division, the integer reappears in the quotient; as in the example of $\frac{4}{2}$, which then become 2.

How may a mixed number, fractionally represented, be reduced? — A mixed number also must reappear as such, by carrying out the division; for the quotient will have a remainder, the denominator not being a measure of the numerator.

How then may a mixed number be reduced fractionally? — *A mixed number is reduced to an improper fraction, by multiplying the integer into the denominator, and adding their product to the numerator.*

Can you exemplify this? — As in the example of a dollar and a half, which is equal to three halves. $1\frac{1}{2} = \frac{3}{2}$.

Nature of Demonstration.

What means the verb, to demonstrate? — To demonstrate is to explain, to show convincingly, to prove.

What is a principle? — *A principle is a truth so clear, as instantly to be admitted on its annunciation; a truth also once clearly proved, and never afterward reasonably to be disputed.*

Can you instance the first kind? — A whole is equal to all its parts.

The second kind? — That division is the solution of multiplication is an example of principles constantly admitted after proof.

In how many ways have the properties of numbers been explained to you? — The properties of numbers have been explained by reasoning from principle and example.

Then what is a demonstration? — *A demonstration is a proof, from principle and example, or from either, of the truth of our conclusions, and the correctness of our operations.*

Greatest common measure.

What is the greatest common measure? — *The greatest common measure of two or more numbers is the largest that will divide all without remainder.*

If one number divide another without remainder; what will the divisor be? — Of two or more numbers any one that divides the rest without remainder, is the greatest common measure of all; for none greater can divide the divisor.

A remainder which measures the divisor is a submultiple of what? — A remainder which measures the divisor is a submultiple of it; for, with some other factor, it produces the divisor.

What other term will such a remainder measure? — A remainder which measures the divisor will measure the whole dividend.

How is this demonstrated? — A submultiple of the divisor measures any multiple of the same; the dividend, exclusive of the remainder, is a multiple of divisor and quotient; the remainder measures itself; therefore, if a remainder measure the divisor, it will measure the whole dividend.

What is the inference under this head? — *A remainder that measures the divisor is the greatest common measure of divisor and dividend.*

Suppose a remainder not to measure the divisor, but that a second remainder measures the first; what will the second be? — A remainder from a divisor made a new dividend, that measures a former remainder from the former dividend, is the greatest common measure of the first divisor and dividend.

How do you demonstrate this? — The divisor, being made a dividend, is, without the second remainder, a multiple of the first remainder, made its divisor; the second remainder measures itself; therefore, if it measure the first, it measures divisor and dividend, and is the greatest common measure of both.

Can you now show by example what you have demonstrated by principle? — Divide 24 by 9, the remainder is 6; divide 9, the divisor, by 6, the remainder, the second remainder is 3. This second remainder measures 6,

$9 : 24 = 2\frac{6}{9}$

$6 : 9 = 1\frac{3}{6}$

$3 : 6 = 2$; therefore,

$3 : 9 = 3$

$3 : 24 = 8$

the first remainder; 9, the divisor; and 24, the dividend; therefore 3, the second remainder, is the greatest common measure of divisor and dividend.

Having found the measure of two numbers what will be the greatest common to it and a third number? — *The last measure obtained in a succession of numbers, by making every preceding measure, as it arises, the divisor of a number yet undivided, is the greatest common measure of all.*

How is this demonstrated? — The last measure is derived from the next preceding, measures it, and thus, through every intermediate one, measures the first obtained: it will therefore divide whatever preceding measures divide, and measuring itself, is the greatest common measure of all the numbers given.

What is the use of common measures? — Common measures are used to reduce fractions to their least terms, by dividing numerator and denominator.

Can you now recite the rule?

RULE TO FIND THE GREATEST COMMON MEASURE.

The greatest common measure of two numbers is found by trying different tabular factors, for such as are common to both. Or,

Divide the greater number by the less; if there be no remainder, the less is the greatest common measure; if there be a remainder, divide the divisor by it, and make any succeeding remainder a divisor of the remainder preceding; the last perfect divisor is the greatest common measure of the two numbers.

When a common measure of more than two numbers is sought, find the greatest measure common to any two of them; then of this common measure and a third number, so proceeding to the last; the last common measure found will be the greatest common to all the numbers.

The last part of the rule is of such rare utility, that exemplification is needless; the former parts are, for the present, sufficiently illustrated by examples already given.

Proof of reduction.

What is meant by reversing an operation? — To reverse an operation is to add what we have subtracted, to divide what we have multiplied, and the contrary, so as to come back to the numbers whence we set out, by using the same factors, or the numbers found, or both,

How may the reduction of fractions be proved?—The correctness of a fractional reduction is proved by reversing the operation; for if the reduction be right, the fraction will reappear, on reversal of the operation, in its original form, or in a form equivalent thereto.

Why do you add equivalent form?—Because the least terms of a fraction are of equal value with any larger terms, and may be raised thereto.

How is this shown?—Any larger terms of a fraction are multiples of the least, or the fraction is not the same, the proportions being different.

How is the operation reversed in the case of improper fractions?—Whole and mixed numbers are reduced to improper fractions by multiplication into a denominator; they are restored to their original form by division of the numerators by their denominators.

Of fractions reduced to their least terms?—Fractions are reduced to their least terms by the division of numerator and denominator, and restored by multiplication into the same factor.

Of fractions brought to a common denominator?—Fractions brought to a common denominator by multiplication, are restored by division, using the same factors; or each may, by any factors, be reduced to its least terms; for these are equivalent to the original terms, if the operation be right.

Universally, what is the mode of proof?—Fractional reduction of any kind is proved by the contrary reduction.

Have you learned the rule?

RULE OF FRACTIONAL REDUCTION.

To reduce a fraction to its least terms, divide numerator and denominator by common measures successively, or at once by their greatest common measure; the last quotients are the terms sought. Should numerator or denominator consist of factors, one factor only in a term must be divided by the same measure.

To reduce fractions to a common denominator, first reduce each to its least terms; then, if one denominator be a multiple of all the other denominators, multiply both terms of their fractions into the corresponding submultiples; if not, multiply all the denominators, one into another, for a common denominator; and each numerator respectively into every denominator except its own, for a proportional numerator.

To reduce a whole number to a fractional form, multiply it into the proposed denominator, and make their product the numerator.

To reduce a mixed number to an improper fraction, multiply the whole number into the denominator of the fraction, and add their product to the numerator.

To reduce an improper fraction, divide numerator by denominator; the quotient will be the equivalent whole or mixed number.

Proof. Reverse the operation, multiplying by former divisors, and dividing by former multipliers; if the reversed operation bring out the original form, the reduction is right. Reduce fractions brought to a common denominator, each to its least terms; if these be equivalent to the original terms, the reduction, in this case also, is right.

APPLICATION.

1. Reduce $\frac{8}{60}$ to its least terms. The example and operation are thus recited: after reading the terms, say, 8 and 60 are both even numbers, are therefore divisible by 2, and 8 are measured by 4; try 4: 4s in 8, two; 4s in 6 one; in 20, five. The fraction $\frac{8}{60} = \frac{4}{15}$. Proof obvious.

2. Reduce $\frac{21}{129}$ to its least terms. Three 7s are 21; try 3; 3s in 21, seven; 3s in 12, four; in 9, three. The fraction $\frac{21}{129} = \frac{7}{43}$. Proof obvious.

3. Reduce $\frac{4}{5}$ and $\frac{6}{25}$ to a common denominator. Twenty-five are a multiple of 5, the least denominator. Five 5s are 25, the common denominator; five 4s are 20, the proportional numerator; therefore $\frac{4}{5} = \frac{20}{25}$.

4. Reduce $\frac{9}{10}$, $\frac{6}{8}$, and $\frac{5}{7}$ to a common denominator. First, the fraction $\frac{6}{8}$ may be reduced to $\frac{3}{4}$, its least terms. Fractions, $\frac{9}{10}$, $\frac{3}{4}$, $\frac{5}{7}$. The denominators, $10 \times 4 \times 7 = 280$, the common denominator.

The numerator 9, multiplied into the denominators 4 & 7	$9 \times 4 \times 7 = 252$	propor. numerator of $\frac{9}{10}$
The numerator	$3 \times 10 \times 7 = 210$	$\frac{3}{4}$
The numerator	$5 \times 10 \times 4 = 200$	$\frac{5}{7}$

Therefore $\frac{9}{10} = \frac{252}{280}$. $\frac{6}{8} = \frac{210}{280}$. $\frac{5}{7} = \frac{200}{280}$. answer.

Proof. $\frac{252}{280}$ are even in both terms; try 4. Fours in 25, six; in 12, three. Fours in 280, seventy. Therefore $\frac{252}{280} = \frac{63}{70} = \frac{9}{10}$, seven being a common measure of both terms. $\frac{210}{280} = \frac{21}{28}$, for both terms may be divided by 10; and $\frac{21}{28} = \frac{3}{4}$, for both terms are measured by 7. Lastly, $\frac{200}{280} = \frac{20}{28}$, by cutting off a cipher from each term; and $\frac{20}{28} = \frac{5}{7}$, 4 being their *common measure.*

5. Reduce 5 to thirds. Three being the proposed denominator, and 3 fives being 15, the integer 3 = $\frac{15}{3}$. Proof obvious.

6. Reduce $7\frac{5}{6}$ to an improper fraction. Multiplying the denominator into the whole number, six 7s are 42; added to 5, are 47; therefore the mixed number, $7\frac{5}{6} = \frac{47}{6}$. Proof obvious.

The proof in the following examples need not be a formal one, where the case is perfectly simple.

Examples to be wrought, proved, and recited.

1. Reduce $\frac{2}{4}$, $\frac{2}{6}$, $\frac{2}{8}$, $\frac{2}{10}$, $\frac{2}{12}$, $\frac{2}{14}$, $\frac{2}{16}$, $\frac{2}{18}$, $\frac{2}{20}$, $\frac{3}{6}$, $\frac{3}{9}$, $\frac{3}{12}$, $\frac{3}{15}$, $\frac{3}{18}$, $\frac{3}{21}$, $\frac{3}{24}$, each to its lowest terms.

2. Reduce $\frac{4}{8}$, $\frac{4}{12}$, $\frac{4}{16}$, $\frac{4}{20}$, $\frac{4}{24}$, $\frac{5}{10}$, $\frac{5}{20}$, $\frac{5}{25}$, $\frac{6}{12}$, $\frac{6}{24}$, $\frac{6}{36}$, $\frac{7}{14}$, $\frac{7}{21}$, $\frac{7}{28}$, $\frac{7}{35}$, each to its lowest terms.

3. Reduce $\frac{8}{16}$, $\frac{8}{24}$, $\frac{8}{32}$, $\frac{9}{18}$, $\frac{9}{27}$, $\frac{9}{36}$, $\frac{10}{20}$, $\frac{11}{22}$, $\frac{11}{33}$, $\frac{11}{44}$, $\frac{12}{24}$, $\frac{12}{48}$, $\frac{13}{26}$, $\frac{13}{39}$, $\frac{13}{52}$, each to its lowest terms.

4. Reduce $\frac{6}{9}$, $\frac{6}{15}$, $\frac{6}{21}$, $\frac{8}{20}$, $\frac{8}{28}$, $\frac{8}{36}$, $\frac{10}{15}$, $\frac{10}{25}$, $\frac{10}{35}$, $\frac{12}{26}$, $\frac{12}{30}$, $\frac{12}{44}$, $\frac{9}{15}$, $\frac{9}{21}$, $\frac{9}{24}$, each to its lowest terms.

5. Prove that the following reductions to least terms are correct, or otherwise: $\frac{144}{240}=\frac{3}{5}$; $\frac{48}{272}=\frac{3}{17}$; $\frac{192}{576}=\frac{1}{3}$; $\frac{825}{960}=\frac{55}{64}$; $\frac{252}{364}=\frac{9}{13}$; $\frac{5184}{6912}=\frac{3}{4}$; $\frac{1344}{1536}=\frac{7}{8}$.

6. Reduce 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, each, to halves, thirds, fourths, and fifths.

7. Reduce $1\frac{1}{2}$, $2\frac{2}{3}$, $3\frac{1}{2}$, $4\frac{3}{5}$, $5\frac{1}{4}$, $6\frac{2}{7}$, $7\frac{5}{6}$, $8\frac{2}{9}$, $9\frac{1}{10}$, $10\frac{2}{9}$, each mixed number to an improper fraction.

8. Reduce $\frac{1}{4}$ and $\frac{1}{2}$, $\frac{2}{3}$ and $\frac{3}{9}$, $\frac{5}{6}$ and $\frac{4}{7}$, $\frac{1}{8}$ and $\frac{9}{10}$, $\frac{3}{6}$ and $\frac{7}{8}$, $\frac{5}{10}$ and $\frac{4}{5}$, $\frac{3}{7}$ and $\frac{2}{11}$, each pair to the least common denominator.

9. Prove that the following reductions to a common denominator are correct, or otherwise: $\frac{1}{2}, \frac{2}{3}, \frac{5}{6}=\frac{3}{6}, \frac{4}{6}, \frac{5}{6}$; $\frac{7}{12}, \frac{11}{18}=\frac{21}{36}, \frac{22}{36}$; $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{5}{6}=\frac{6}{12}, \frac{8}{12}, \frac{9}{12}, \frac{10}{12}$; $\frac{2}{5}, \frac{4}{6}, \frac{5}{9}, \frac{7}{10}, =\frac{36}{90}, \frac{60}{90}, \frac{50}{90}, \frac{63}{90}$.

All the above reductions, with the exception of those in divisions 5 and 9, should be made mentally, on interrogation, after the whole series has been wrought on the slate.

Addition.

How are whole numbers added to fractions? — Whole numbers are added to fractions by prefixing their sum.

When, by the addition of fractions, an improper fraction is obtained, how should the sum total be represented? — An improper fraction arising from addition should be reduced to a whole or mixed number.

When a sum total may be reduced to smaller terms, what should be done? — Fractions should always be reduced to their least terms, when there exists no reason for leaving them in a larger form.

How is the addition of fractions proved? — The number of common fractions added together is usually too small to require any thing more than attention in the procedure; they may, however, be reckoned in a reversed direction, and should always be so when a column consists of fractions of a cent or of a penny. The reductions are proved by their proper rule.

Do you recollect the maxim?

Maxim in the addition of Fractions.

Fractions are added by their numerators, their denominators being the same, originally or by reduction.

APPLICATION.

What is the sum total of $\frac{4}{5}$ and $\frac{7}{8}$? Five 8s are 40, the common denominator of these fractions. Four 8s are 32, proportional numerator of $\frac{4}{5}$; seven 5s are 35, proportional numerator of $\frac{7}{8}$. Thirty-two 40ths added to $\frac{35}{40}$ $=\frac{67}{40}$, by division equal to the mixed number, $1\frac{27}{40}$.

$\frac{4}{5}+\frac{7}{8}$.
$5\times8=40$ com. denom.
$4\times8=32$ p. num. of $\frac{4}{5}$.
$7\times5=35$ p. num. of $\frac{7}{8}$.
$\frac{32}{40}+\frac{35}{40}=\frac{67}{40}$.
$40 : 67=1\frac{27}{40}$ ans.

Examples to be wrought and recited.

What is the sum of

1. $\frac{2}{3}+\frac{1}{3}$? $\frac{3}{4}+\frac{1}{4}$? $\frac{2}{5}+\frac{3}{5}$? $\frac{1}{6}+\frac{4}{6}$? $\frac{3}{7}+\frac{2}{7}$? $\frac{5}{8}+\frac{6}{8}$? $\frac{4}{9}+\frac{5}{9}$? $\frac{7}{10}+\frac{9}{10}$?
2. $\frac{1}{2}+\frac{2}{3}$? $\frac{1}{4}+\frac{1}{3}$? $\frac{1}{4}+\frac{4}{5}$? $\frac{5}{6}+\frac{1}{7}$? $\frac{1}{8}+\frac{2}{9}$? $\frac{3}{10}+\frac{1}{2}$? $\frac{4}{10}+\frac{1}{6}$? $\frac{1}{11}+\frac{7}{9}$?
3. $\frac{2}{3}+\frac{5}{6}$? $\frac{3}{4}+\frac{6}{7}$? $\frac{4}{5}+\frac{3}{10}$? $\frac{5}{6}+\frac{5}{7}$? $\frac{3}{4}+\frac{5}{10}$? $\frac{4}{7}+\frac{2}{11}$? $\frac{5}{9}+\frac{7}{12}$?
4. $\frac{1}{8}+\frac{5}{12}$? $\frac{6}{8}+\frac{1}{2}$? $\frac{4}{9}+\frac{1}{13}$? $\frac{3}{7}+\frac{2}{13}$? $\frac{5}{6}+\frac{10}{12}$? $\frac{5}{9}+\frac{2}{3}$? $\frac{3}{13}+\frac{7}{8}$?
5. $\frac{3}{11}+\frac{3}{22}$? $\frac{5}{12}+\frac{7}{24}$? $\frac{7}{10}+\frac{9}{20}$? $\frac{4}{7}+\frac{4}{21}$? $\frac{3}{4}+\frac{5}{10}$? $\frac{2}{5}+\frac{7}{25}$? $\frac{8}{9}+\frac{9}{18}$?

It is presumed that the preceding reductions and additions will not be found too difficult for merely mental performance, on interrogation, after having been wrought on the slate and recited.

Prove the correctness, or the contrary, of the following additions:

6. $\frac{3}{4}+\frac{1}{4}+\frac{1}{2}=1\frac{1}{2}$. $\frac{6}{7}+\frac{3}{14}+\frac{5}{7}=1\frac{11}{14}$. $\frac{2}{3}+\frac{1}{6}+\frac{5}{6}=1\frac{2}{3}$.
7. $\frac{3}{10}+\frac{4}{30}+\frac{7}{30}=\frac{2}{3}$. $\frac{5}{6}+\frac{5}{12}+\frac{5}{6}=2\frac{1}{12}$. $\frac{2}{3}+\frac{1}{9}+\frac{5}{18}=1\frac{1}{18}$.
8. $\frac{3}{5}+\frac{7}{10}+\frac{8}{25}=1\frac{31}{50}$. $\frac{3}{8}+\frac{7}{16}+\frac{1}{4}+\frac{1}{2}=1\frac{9}{16}$. $\frac{5}{8}+\frac{7}{8}+\frac{7}{16}=1\frac{15}{16}$.
9. $\frac{3}{5}+\frac{4}{15}+9\frac{3}{20}=10\frac{1}{60}$. $\frac{2}{3}+\frac{5}{9}+\frac{7}{18}+5=6\frac{11}{18}$.
10. $\frac{7}{22}+\frac{9}{11}+\frac{22}{44}=1\frac{7}{11}$. $\frac{5}{12}+\frac{7}{48}+1\frac{5}{96}=1\frac{59}{96}$. $\frac{7}{12}+\frac{5}{9}+\frac{7}{8}=2\frac{1}{72}$.
11. $\frac{9}{13}+\frac{1}{12}+\frac{2}{11}+4=4\frac{1643}{1716}$. $\frac{4}{7}+\frac{7}{9}+\frac{5}{8}=1\frac{491}{504}$. $\frac{8}{9}+2\frac{1}{18}+\frac{5}{6}=3\frac{7}{9}$.
12. $\frac{3}{5}+6\frac{1}{13}+8\frac{2}{15}=15\frac{8}{195}$. $\frac{7}{29}+\frac{5}{41}+\frac{11}{32}=\frac{26903}{38048}$.

Subtraction.

What is the symbol of subtraction?

Do you recollect the maxim in fractional subtraction?

Maxim in the Subtraction of Fractions.

Fractions are subtracted by their numerators, their denominators being the same, originally or by reduction.

From a dollar and a quarter take $\frac{3}{4}$; what remains? — Three quarters, taken from a dollar and a quarter, leave $\frac{1}{2}$.

Can you show this arithmetically? — One dollar and a quarter are equal to $\frac{5}{4}$; from these take $\frac{3}{4}$, and there remain $\frac{2}{4}$, or $\frac{1}{2}$. $1\frac{1}{4}=\frac{5}{4}$. $\frac{5}{4}-\frac{3}{4}=\frac{2}{4}=\frac{1}{2}$.

How is fractional subtraction proved? — Fractional subtraction is proved by fractional addition; reduction by the contrary reduction.

Can you repeat the rule?

RULE FOR THE SUBTRACTION OF FRACTIONS.

To subtract fractions from fractions, reduce those of different denominators to the same, and from the numerator, or sum total of the numerators of the minuend, take the numerator or numerators of the subtractor.

To subtract a fraction from an integer, reduce a unit of the whole number to an improper fraction of the same denominator; subtract by the numerators, and carry 1, to be taken from the integer.

To subtract a fraction from a mixed number, reduce a unit and its fraction, if necessary, into an improper fraction of the same denominator with the subtractor; subtract by the numerators, and carry 1, to be taken from the integer.

Proof. Fractional subtraction is proved by fractional addition.

APPLICATION.

The manner of recitation will no longer be given, it being presumed that you are now able to read the signs by words.

1. From $\frac{3}{4}$ take $\frac{5}{12}$; how much remains?

$\frac{3}{4}=\frac{9}{12}$; and $\frac{9}{12}-\frac{5}{12}=\frac{4}{12}=\frac{1}{3}$, answer.

2. From 29 take $\frac{11}{15}$; what remains?

$1=\frac{15}{15}$; and $28\frac{15}{15}-\frac{11}{15}=28\frac{4}{15}$, answer.

3. From $11\frac{6}{7}$ take $\frac{7}{8}$; what is the difference?

$7\times8=56$ com. denominator.
$6\times8=48$ prop'l. num. of $\frac{6}{7}$
$7\times7=49$ prop'l. num. of $\frac{7}{8}$

(11 or) $10\frac{56}{56}+\frac{48}{56}=10\frac{104}{56}$ min.

$10\frac{104}{56}-\frac{49}{56}=10\frac{55}{56}$, answer.

4. How much greater are $1206\frac{3}{19}$ than $455\frac{19}{21}$?

$1205\frac{462}{399}$	$19\times21=399$ com. denominator.
$455\frac{361}{399}$	$3\times21=\ 63$ prop'l. numer. of $\frac{3}{19}$
———	$19\times19=361$ prop'l. numer. of $\frac{19}{21}$
$750\frac{101}{399}$ answer.	$399+63=462$ p. num. of imp. frac.

The multiplications that cannot be made *memoriter* must be done aside, and their results placed in line.

	$63 : 399=6$	$21 : 399=19$
$750\frac{101}{399}$	378	21
$455\frac{361}{399}$	——	——
———	G. com. m. $21 : 63=3$	189
$1206\frac{63}{399}$ proof in larger terms.		189

Therefore $1206\frac{63}{399}=1206\frac{3}{19}$ proof.

Examples to be wrought and recited.

What remains from

1. $\frac{1}{2}-\frac{1}{2}$? $\frac{3}{4}-\frac{1}{4}$? $\frac{3}{5}-\frac{2}{5}$? $\frac{5}{6}-\frac{1}{6}$? $\frac{5}{7}-\frac{3}{7}$? $\frac{7}{8}-\frac{5}{8}$? $\frac{7}{9}-\frac{2}{9}$? $\frac{9}{10}-\frac{7}{10}$?
2. $\frac{1}{2}-\frac{1}{4}$? $\frac{1}{4}-\frac{1}{8}$? $\frac{1}{5}-\frac{1}{10}$? $\frac{1}{6}-\frac{1}{12}$? $\frac{1}{7}-\frac{1}{14}$? $\frac{1}{8}-\frac{1}{16}$? $\frac{1}{9}-\frac{1}{18}$?
3. $\frac{2}{5}-\frac{1}{10}$? $\frac{5}{6}-\frac{7}{12}$? $\frac{5}{7}-\frac{3}{14}$? $\frac{7}{8}-\frac{5}{16}$? $\frac{5}{9}-\frac{7}{18}$? $\frac{9}{10}-\frac{2}{5}$? $\frac{1}{2}-\frac{1}{3}$?
4. $\frac{3}{16}-\frac{1}{8}$? $\frac{8}{15}-\frac{2}{5}$? $\frac{3}{14}-\frac{1}{7}$? $\frac{11}{12}-\frac{2}{3}$? $\frac{7}{12}-\frac{1}{4}$? $\frac{5}{9}-\frac{1}{3}$? $\frac{3}{4}-\frac{3}{5}$?
5. $\frac{3}{4}-\frac{5}{9}$? $\frac{1}{7}-\frac{1}{6}$? $\frac{3}{7}-\frac{2}{5}$? $\frac{4}{7}-\frac{3}{8}$? $\frac{3}{8}-\frac{1}{9}$? $\frac{5}{8}-\frac{3}{7}$? $\frac{7}{8}-\frac{3}{10}$? $\frac{1}{8}-\frac{1}{11}$?
6. $\frac{3}{8}-\frac{1}{12}$? $\frac{7}{8}-\frac{5}{12}$? $\frac{5}{9}-\frac{3}{10}$? $\frac{7}{9}-\frac{2}{11}$? $\frac{4}{9}-\frac{5}{12}$? $\frac{5}{9}-\frac{2}{7}$? $\frac{8}{9}-\frac{6}{7}$?
7. $\frac{1}{10}-\frac{1}{11}$? $\frac{3}{10}-\frac{1}{12}$? $\frac{5}{10}-\frac{2}{9}$? $\frac{7}{10}-\frac{5}{8}$? $\frac{9}{10}-\frac{8}{9}$? $\frac{9}{10}-\frac{9}{11}$?
8. $\frac{5}{11}-\frac{5}{12}$? $\frac{6}{11}-\frac{1}{2}$? $\frac{7}{11}-\frac{3}{7}$? $\frac{9}{11}-\frac{7}{12}$? $1-\frac{1}{3}$? $1-\frac{3}{4}$? $1-\frac{4}{5}$?
9. $1-\frac{3}{7}$? $1-\frac{5}{8}$? $2-\frac{1}{2}$? $2-1\frac{1}{2}$? $2-1\frac{3}{5}$? $2\frac{1}{2}-1\frac{1}{4}$? $2\frac{1}{4}-\frac{3}{4}$?

The above are for mental performance also on interrogation.

Prove that the differences stated below are correct, or the contrary.

10. $\frac{5}{8}-\frac{1}{4}=\frac{3}{8}$. $\frac{7}{9}-\frac{1}{6}=\frac{33}{54}$. $\frac{4}{7}-\frac{1}{3}=\frac{5}{21}$. $\frac{6}{11}-\frac{1}{2}=\frac{1}{22}$. $\frac{5}{6}-\frac{7}{9}=\frac{1}{18}$.
11. $\frac{2}{7}-\frac{2}{9}=\frac{4}{63}$. $\frac{5}{12}-\frac{2}{7}=\frac{11}{84}$. $\frac{1}{5}-\frac{1}{8}=\frac{3}{40}$. $\frac{10}{11}-\frac{10}{13}=\frac{20}{143}$.
12. $\frac{3}{14}-\frac{1}{10}=\frac{4}{35}$ $\frac{2}{15}-\frac{1}{12}=\frac{1}{20}$. $\frac{7}{16}-\frac{3}{7}=\frac{1}{112}$. $\frac{6}{17}-\frac{2}{9}=\frac{20}{153}$.
13. $\frac{5}{7}-\frac{9}{20}=\frac{37}{140}$. $\frac{7}{9}-\frac{10}{21}=\frac{57}{189}$. $\frac{3}{5}-\frac{12}{35}=\frac{9}{35}$. $\frac{6}{7}-\frac{3}{43}=\frac{237}{301}$.
14. $1\frac{2}{5}-\frac{3}{4}=\frac{13}{20}$. $2\frac{5}{6}-\frac{7}{8}=1\frac{23}{24}$. $3\frac{4}{9}-\frac{11}{12}=2\frac{57}{108}$. $7\frac{3}{4}-\frac{9}{10}=6\frac{17}{20}$.
15. $1\frac{3}{8}-1\frac{1}{5}=\frac{7}{40}$. $9\frac{4}{11}-6\frac{7}{9}=2\frac{58}{99}$. $7-6\frac{1}{9}=\frac{8}{9}$. $3\frac{4}{13}-2\frac{5}{12}=\frac{129}{156}$.

What remains from

16. $\frac{13}{14}-\frac{9}{12}$? $\frac{1}{20}$ and $\frac{17}{19}-\frac{11}{16}$? $\frac{2}{5}$ and $\frac{17}{25}-\frac{9}{1}$? $\frac{5}{11}$ and $\frac{7}{37}-\frac{15}{44}$?
17. $5\frac{3}{4}-\frac{14}{99}$? $\frac{1}{2}$ and $2\frac{7}{60}-\frac{29}{59}$? $\frac{3}{4}$ and $8\frac{5}{9}-3\frac{36}{75}$? $17\frac{3}{4}-11\frac{45}{61}$?

Multiplication

Of fractions by integers.

How are we to understand the term multiply, in its application to fractions? — To *multiply*, used of a fractional multiplier, must be understood in the sense of taking; since a fractional multiplier lessens the multiplicand, by taking only a part of it.

Does this constantly affect the result? — In multiplication by a fraction, the product is, in every case, less than the multiplicand.

In multiplication by integers how is it? — Multiplication by integers produces always a sum equal to or greater than the multiplicand; for either the whole is taken, or more than the whole.

Which is the diminishing term of a fraction? — The denominator of a fraction is diminuent, for it is a divisor.

How then would you multiply a fraction by an integer? — In the multiplication of fractions by whole numbers, the numerator must be the term multiplied; for multiplication of the denominator would be diminution of the multiplicand.

Can you give me an example? — A quarter of a dollar, multiplied by the number 4, produces a dollar, the numerator becoming equal to the denominator. $\frac{1}{4}\times4=\frac{4}{4}=1.$

What is the effect of dividing a denominator? — To divide a denominator, or to multiply a numerator, by the same factor, gives the same result.

Can you exemplify this also? — The denominator of a quarter, if divided by 4, leaves the fraction equal to unity. $\frac{1}{4:4}=\frac{1}{1}=1.$

When would you prefer dividing the denominator? — I would divide the denominator, rather than multiply the numerator, whenever the former could be done without leaving a remainder.

Hence what rule arises? — *To multiply a fraction by a whole number, multiply it into the numerator, or by it divide the denominator.*

Of mixed by whole or mixed numbers.

Of two factors, one a mixed, the other a whole number, which would you make the multiplier? — In a case of numerous places, unless the significants of the mixed number should much exceed those of the pure factor, I should prefer making the mixed factor the multiplier.

How would you begin? — I would first multiply the entire integral multiplicand into the numerator of the fraction; if the product should be a proper fraction, I would set down the fractional parts; if improper, I would reduce it, and carry the units to the next partial product; or if too large to be thus carried, I would make of the fractional a first separate product.

What advantages are derived from this method? — By taking the fractional product first, we follow the order of the factors; by taking it at once, the serious inconvenience of a number of small fractions, of different denominators, and of different places, is avoided.

Should both factors be mixed numbers, how would you proceed? — With two mixed factors, I would first take the whole multiplicand by the fraction of the multiplier, for the first separate product; I would then take the whole multiplier, except the fraction, by the fraction of the multiplicand, for the second separate product.

Why would you except the fraction of the multiplier? — The fractions of both factors being multiplied into each other on the first multiplication, must not be so a second time.

Of whole or mixed numbers by fractions.

How would you multiply a whole number by a fraction? — A whole number is multiplied by a fraction as a fraction is multiplied by a whole number, by multiplying it into the numerator; dividing then their product, if improper, by the denominator, we obtain the entire product.

Suppose you divide the denominator? — As in the same case; to divide the denominator by the whole number, or to multiply the numerator, gives the same result.

Can you again exemplify these methods? — Eight dollars, multiplied into $\frac{3}{4}$, thus appear to produce, what in fact we know they do, six dollars; for eight 3s are 24; and 4 are contained in 24 six times. $8\times\frac{3}{4}=\frac{24}{4}=6$.

That also by division of the denominator? — Twelve and a half cents are an eighth of a dollar; and $\frac{1}{8}$ of a dollar taken 8 times, by division of the denominator, produces an improper fraction equal to a unit, or one dollar. $\frac{1}{8:8}=\frac{1}{1}=1$.

How would you multiply a mixed number by a fraction? — A mixed number, to be multiplied into a fraction, must be multiplied by the numerator, and their product divided by the denominator.

You would make the fraction the multiplier then? — Of two factors, one mixed, the other purely fractional, I would choose a fractional multiplier; as this might require but a single product, and would involve little reduction.

In what manner would you proceed? — I would multiply the whole mixed factor into the numerator of the fractional multiplier, leaving the denominator in the multiplicand unchanged, and by it reducing the first partial and fractional product, if improper; I would then divide the integral product by the denominator of the multiplier, reduce any remainder to the form of the fractional part, and multiply the denominators together.

Of fractions by fractions.

What is the product of the half of half a dollar? — Half a dollar, multiplied into one half, can make no more than a quarter.

How is this shown arithmetically? — Multiplication by a fraction whose numerator is 1, is division by the denominator, and a fraction is divided by multiplying its denominator; therefore I multiply denominator into denominator. $\frac{1}{2}\times\frac{1}{2}=\frac{1}{4}$.

How many pieces of \$.06$\frac{1}{4}$ value are there in a dollar? — In a dollar there are 16 pieces of 6$\frac{1}{4}$ cents.

How many are $\frac{3}{4}$ of those 16? — In $\frac{3}{4}$ of a dollar there are 12 pieces of 6$\frac{1}{4}$ cents; for in each quarter there are four.

Can you show arithmetically how many $\frac{3}{4}$ of those 12 are? — Twelve, the whole number, multiplied into 3, the numerator, produce 36: these divided by 4, the denominator, give 9 to the quotient; and 9 we know to be three quarters of 12. $\frac{3}{4}\times 12=\frac{36}{4}=9$.

Twelve pieces then being equal to $\frac{3}{4}$ of a dollar, can you not take, arithmetically, $\frac{3}{4}$ of $\frac{3}{4}$? — By multiplying numerator into numerator, and denominator into denominator, nine sixteenths of a dollar, or considered as coins, 9 pieces of 6$\frac{1}{4}$ cents each, are obtained. $\frac{3}{4}\times\frac{3}{4}=\frac{9}{16}$.

Assured of the truth of our methods, let us now consider the reason of them. A fraction taken once produces how much? — Taken once only, a fraction remains unaltered.

Taken in part, what will it be? — Taken less than once, it becomes a still smaller fraction.

How are fractions lessened? — Fractions are lessened by increasing their denominators.

Increased by addition, or what? — Increased by multiplication, if the operative term be a factor; for every factor is ex-

pressive of a number of turns, or parts of a turn; turns are a repetition of the same number, which addition, except in cases in which it is identified with multiplication, is not.

By what term of a fractional multiplier shall the denominator of the multiplicand be increased? — The denominator of the multiplicand must be increased by the same term of the multiplier; for the purpose is diminution, and the diminishing term of a fraction is the denominator.

What is the operation when you take one part only of a fraction? — To obtain that one part of a fraction which is denominated by some other fraction, the operation is altogether diminution; for we multiply the denominator; as in taking $\frac{1}{2}$ of $\frac{1}{2}$, $\frac{1}{2}$ of $\frac{3}{4}$.

Might you not divide the numerator? — To divide the numerator lessens a fraction equally with multiplying the denominator; for the office of the denominator itself is to divide.

What is the process in taking more parts than one? — To obtain more parts than one of a fraction, parts, namely, denominated from some other fraction, there must also be an operation of increase, or more would be the same with one; and fractions are increased by multiplication of their numerators or division of their denominators.

By what term of a fractional factor would you increase them? — Their numerators must be increased by multiplication into the same terms of the other factors; for numerators are terms of increase, by expressing a number of terms which denominators diminish to parts of a turn.

Can you exemplify this case? — In taking $\frac{3}{4}$ of $\frac{3}{4}$ of a dollar, by multiplication of the denominators together, we take a fourth part of three fourths, or three sixteenths: these, by multiplication of the numerators together, are increased to three times three, or nine, sixteenths.

Hence what rule of multiplication may be formed? — *To multiply fractions by fractions, multiply numerator into numerator, and denominator into denominator.*

Discrimination of factors.

How may fractional factors in multiplication be known? — *Every fraction denominated a part of some other number is a multiplier of that number.*

Wherefore? — Because part of a number is taken by multiplication.

Not by division also? — Division does not take parts, it *mbers them* or values them.

Suppose the question to be, what part is such a fraction? — The question, what part of any number is such a fraction, shows that the part is still unknown, therefore not yet numbered, or denominated; and to give a denomination to equal parts, is the office of division.

Complex fractions.

What are the fractions usually termed compound? — Compound fractions, so called, are really fractional factors in multiplication; as $\frac{1}{2}$ of $\frac{1}{3}$.

What are complex fractions? — Complex fractions are such as have other fractions in numerator or denominator, or in both.

Can you give me an example? — One and a half divided by three ($\frac{1\frac{1}{2}}{3}$) is an example.

What is the proportion of those terms to each other? — One and a half are precisely the half of 3.

What is the effect of multiplying both terms of a fraction? — To multiply numerator and denominator by the same factor, leaves the fraction unaltered in value.

May not a complex fraction be thus simplified? — If both terms be multiplied into the denominator of any fraction the terms may contain, the fraction will be simplified, and its value unchanged.

How do you exemplify it in the case alleged? — By multiplying the numerator of $\frac{1\frac{1}{2}}{3}$ into 2, its fraction becomes a unit, and the integer, by multiplication and addition, is increased to 3; the denominator to 6; the fraction is then $\frac{3}{6}$ or $\frac{1}{2}$; being of the same value with the complex form.

Supposing both terms to be complex, how would you clear the fraction? — Should both terms be complex, one may be cleared after the other; or both terms may be cleared at once, by multiplication of both into the product of the complicating denominators.

Can you exemplify this case? — If the numerator be $2\frac{1}{8}$, the denominator, $12\frac{3}{4}$, the factor, 8, (it so happening,) simplifies both terms, which, by reduction to the least, become $\frac{1}{6}$.

$$\frac{2\frac{1}{8} \times 8}{12\frac{3}{4} \times 8} = \frac{17}{102} = \frac{1}{6}.$$

Can you now repeat the maxim on this subject?

Maxim in the simplification of complex fractions.

Complex fractions are simplified by multiplication of both terms into *whatever* complicating denominators they may *contain.*

Consideration of proof.

How is the multiplication of fractions proved? — Fractional multiplication must be proved by fractional division.

Multiply a mixed number by the denominator of its fraction; what is the result? — The multiplication of a mixed number into the denominator of its fraction will occasion the disappearance of the fraction itself.

Whence this circumstance? — The multiplication of a fraction into its denominator is equivalent to the removal of the denominator; the numerator, therefore, becoming a whole number, is comprehended within the product by addition.

If divisor and dividend be multiplied into the same number, will this affect the quotient? — The quotient is unaffected by an equal multiplication of divisor and dividend; for their proportions are unchanged.

Can you now recite the rule entire?

RULE OF FRACTIONAL MULTIPLICATION.

To multiply a whole number into a fraction, multiply it into the numerator, and divide their product by the denominator; or, if it measure the denominator, divide the denominator by the whole number.

To multiply fractions together, multiply numerator into numerator, and denominator into denominator; the resulting fraction is the product.

Proof. Fractional multiplication is proved by fractional division. To prove the multiplication of mixed numbers, proceed as in the proof of integral multiplication. If one factor be a whole number, make that the divisor; if both factors be mixed numbers, simplify one for a divisor, by multiplying it into its denominator; multiply the entire product by the same; if the work be right, the quotient will be the other factor, or its equivalent.

N. B. Equivalent, because any such fraction will be in larger terms, produced by the simplification of the divisor; reduced by some common measure, it will appear in its proper terms.

APPLICATION.

1. What is the product of $155 \times 78\frac{2}{3}$?

$$\begin{array}{l} 155 \\ \quad 78\frac{2}{3} \\ \hline 3 : 310 \\ \hline \quad 103\frac{1}{3} \\ \quad 1240 \\ 1085 \\ \hline 12193\frac{1}{3} \text{ answer.} \end{array}$$

$$\begin{array}{l} 155 : 12193\frac{1}{3} = 78\frac{2}{3} \text{ proof.} \\ \quad 1085 \\ \hline \quad 1343\frac{1}{3} \\ \quad 1240 \\ \hline \quad 103\frac{1}{3} : 155 = 1 \\ \qquad 103\frac{1}{3} \\ \hline \text{G. com. meas. } 51\frac{2}{3} : 103\frac{1}{3} = 2 \\ \qquad 103\frac{1}{3} \end{array} \qquad \begin{array}{l} 51\frac{2}{3} : 155 = 3 \\ \qquad 155 \end{array}$$

We first take $\frac{2}{3}$ of the multiplicand: in order to this, we multiply it by the numerator of the mixed multiplier, and divide their product by the denominator; the quotient is the first separate product. The proof, on division, leaves the complex fraction $\frac{103\frac{1}{3}}{155}$; the greatest common measure of these two numbers appears, on division by the second remainder, to be $51\frac{2}{3}$, reducing the complex fraction derived from the remainder to $\frac{2}{3}$, the fraction in its original form. In finding the greatest common measure, with a view to the proof of mixed or fractional multiplications, it is never necessary to simplify remainders; for if the multiplication be correct, the first or last remainder will certainly measure the divisor; as in the example above. When a dividend contains a fraction, it should be brought down with the last place, as it constitutes no place of itself, and may otherwise be overlooked.

2. What is the product of $906\frac{2}{9} \times 315\frac{4}{11}$?

$$\begin{array}{l} 315\frac{4}{11} \qquad \overline{906 + \frac{4}{11}} \\ 906\frac{2}{9} \quad 11 : 3624 = 329\frac{5}{11} \\ \hline 9 : 630\frac{8}{11} \\ \hline \quad 70\frac{8}{99} \\ \quad 329\frac{45}{99} \qquad \frac{5}{11} = \frac{45}{99} \\ \quad 1890 \\ 2835 \\ \hline 285789\frac{53}{99} \text{ answer.} \end{array}$$

$$\begin{array}{l} 906\frac{2}{9} \quad 285789\frac{53}{99} \\ \quad 9 \qquad\quad 9 \\ \hline 8156 : 2572105\frac{9}{11} = 315\frac{2965\frac{9}{11}}{8156} \\ 2965\frac{9}{11} : 8156 = 2 \\ \qquad 5931\frac{7}{11} \\ \hline \qquad 2224\frac{4}{11} : 2965\frac{9}{11} = 1 \\ \qquad\qquad 2224\frac{4}{11} \\ \hline \text{Grea. com. meas. } 741\frac{5}{11} : 2224\frac{4}{11} = 3 \\ 741\frac{5}{11} : \overline{2965\frac{9}{11}} = 4 \\ 741\frac{5}{11} : \overline{8156} = \overline{11} \text{ proof.} \end{array}$$

In this example of two mixed factors, we are under the necessity of operating on the integral part of the multiplier, by the fraction

of the multiplicand, aside ; the product obtained is then inserted in its right place under the first separate product, having its fraction reduced to a denominator common to the fraction with which it is to be added up. In the proof, because it is extremely difficult, except in the case of a common measure sought in proof, to operate with a mixed divisor, we have simplified that chosen for the divisor. The products and subtractions are omitted, as being easily understood ; but the process of obtaining the common measure is given, that from these two examples, the learner may have a clear comprehension of its nature.

Examples to be wrought and recited.

What is the product of

1. $\frac{1}{2}$ of 3? of 5? of 7? of 9? $\frac{1}{3}$ of 2? of 3? of 9? $\frac{1}{4}$ of 2? of 3? of 4?

2. $\frac{1}{4}$ of 5? of 6? of 7? of 9? $\frac{1}{5}$ of 2? of 3? of 4? of 5? of 6? of 7? of 9?

3. $\frac{2}{3}$ of 2? of 3? of 4? of 5? of 6? of 7? of 8? $\frac{3}{4}$ of 2? of 3? of 4?

4. $\frac{3}{4}$ of 5? of 6? of 7? of 8? $\frac{2}{5}$ of 2? of 3? $\frac{4}{5}$ of 4? $\frac{2}{5}$ of 6? $\frac{2}{5}$ of 7? of 8?

5. $\frac{4}{5}$ of 8? $\frac{2}{5}$ of 9? $\frac{5}{6}$ of 2? of 3? $\frac{1}{6}$ of 4? of 5? $\frac{5}{6}$ of 7? of 8? of 9?

6. $\frac{2}{7}$ of 2? of 3? of 4? $\frac{3}{7}$ of 5? $\frac{4}{7}$ of 6? $\frac{5}{7}$ of 8? $\frac{6}{7}$ of 9? $\frac{3}{8}$ of 2? of 4?

7. $\frac{1}{8}$ of 3? $\frac{5}{8}$ of 5? of 6? $\frac{7}{8}$ of 7? of 9? $\frac{2}{9}$ of 2? $\frac{4}{9}$ of 3? $\frac{1}{9}$ of 4? $\frac{5}{9}$ of 5?

8. $\frac{7}{9}$ of 6? $\frac{8}{9}$ of 7? $\frac{5}{9}$ of 9? $\frac{3}{10}$ of 2? of 3? $\frac{7}{10}$ of 4? $\frac{1}{10}$ of 5? $\frac{3}{10}$ of 6? $\frac{7}{10}$ of 7?

9. $\frac{9}{10}$ of 8? $\frac{2}{11}$ of 2? $\frac{3}{11}$ of 3? $\frac{1}{11}$ of 4? $\frac{4}{11}$ of 5? $\frac{5}{11}$ of 6? $\frac{6}{11}$ of 7? $\frac{7}{11}$ of 9?

10. $\frac{5}{12}$ of 2? $\frac{1}{12}$ of 3? $\frac{7}{12}$ of 4? $\frac{9}{12}$ of 5? $\frac{1}{13}$ of 6? $\frac{2}{13}$ of 7? $\frac{3}{13}$ of 8? $\frac{4}{13}$ of 9?

11. $\frac{1}{2}$ of $\frac{1}{2}$? of $\frac{1}{4}$? of $\frac{1}{5}$? of $\frac{1}{6}$? of $\frac{1}{7}$? of $\frac{1}{8}$? of $\frac{1}{9}$? $\frac{1}{3}$ of $\frac{1}{2}$? $\frac{1}{4}$ of $\frac{1}{3}$?

12. $\frac{1}{5}$ of $\frac{1}{4}$? $\frac{1}{6}$ of $\frac{1}{5}$? $\frac{1}{7}$ of $\frac{1}{6}$? $\frac{1}{8}$ of $\frac{1}{7}$? of $\frac{1}{9}$? $\frac{2}{3}$ of $\frac{1}{2}$? $\frac{3}{4}$ of $\frac{1}{3}$? $\frac{4}{5}$ of $\frac{1}{4}$?

13. $\frac{5}{6}$ of $\frac{1}{5}$? $\frac{6}{7}$ of $\frac{1}{6}$? $\frac{7}{8}$ of $\frac{1}{7}$? $\frac{8}{9}$ of $\frac{1}{9}$? $\frac{3}{5}$ of $\frac{1}{2}$? of $\frac{1}{3}$? $\frac{2}{7}$ of $\frac{1}{4}$? $\frac{3}{7}$ of $\frac{1}{5}$?

14. $\frac{5}{7}$ of $\frac{1}{6}$? $\frac{3}{8}$ of $\frac{1}{7}$? $\frac{5}{8}$ of $\frac{1}{8}$? $\frac{3}{4}$ of $\frac{3}{4}$? of $\frac{2}{5}$? of $\frac{3}{5}$? of $\frac{4}{5}$? of $\frac{5}{6}$? of $\frac{2}{7}$?

15. $\frac{2}{5}$ of $\frac{5}{6}$? of $\frac{3}{7}$? of $\frac{5}{7}$? of $\frac{6}{7}$? of $\frac{3}{8}$? of $\frac{5}{8}$? of $\frac{4}{9}$? $\frac{3}{5}$ of $\frac{5}{6}$? of $\frac{3}{7}$? of $\frac{5}{7}$?

16. $\frac{3}{5}$ of $\frac{6}{7}$? of $\frac{3}{8}$? of $\frac{5}{8}$? of $\frac{4}{9}$? $\frac{5}{8}$ of $\frac{4}{5}$? of $\frac{4}{7}$? of $\frac{5}{7}$? of $\frac{6}{7}$? of $\frac{3}{8}$? of $\frac{5}{8}$?

17. $\frac{5}{6}$ of $\frac{1}{9}$? $\frac{3}{4}$ of $\frac{2}{9}$? $\frac{3}{8}$ of $\frac{3}{10}$? $\frac{5}{9}$ of $\frac{4}{11}$? $\frac{7}{10}$ of $\frac{6}{11}$? $\frac{8}{11}$ of $\frac{5}{12}$? $\frac{7}{12}$ of $\frac{2}{13}$? $\frac{4}{13}$ of $\frac{3}{13}$?

The preceding examples are for mental solution also, on interrogation.

18. $\frac{11}{20}$ of $\frac{9}{15}$? $\frac{12}{31}$ of $\frac{16}{17}$? $\frac{13}{35}$ of $\frac{12}{54}$? $\frac{18}{19}$ of $\frac{17}{18}$? $\frac{20}{35}$ of $\frac{15}{90}$? $\frac{72}{144}$ of $\frac{22}{121}$?

19. $54\times\frac{9}{19}$? $76\times\frac{5}{17}$? $104\times\frac{9}{18}$? $361\times\frac{75}{175}$? $560\times\frac{91}{100}$? $718\times\frac{78}{96}$? $515\times\frac{63}{93}$?

20. $104\times9\frac{3}{8}$? $706\times28\frac{7}{9}$? $393\times42\frac{6}{7}$? $811\times17\frac{3}{17}$? $908\times73\frac{13}{15}$? $691\times354\frac{24}{41}$?

21. $803\times288\frac{76}{104}$? $718\times302\frac{64}{97}$? $237\times564\frac{78}{144}$? $98\times97\frac{13}{19}$? $230\frac{3}{4}\times109\frac{5}{8}$?

22. $617\frac{9}{10}\times111\frac{33}{100}$? $526\frac{13}{14}\times717\frac{55}{200}$? $110\frac{12}{13}\times444\frac{7}{13}$? $791\frac{77}{107}\times531\frac{19}{98}$?

23. $323\frac{16}{24}\times991\frac{17}{51}$? $168\frac{37}{54}\times731\frac{7}{8}$? $810\frac{69}{72}\times198\frac{3}{34}$? $459\frac{1}{2}\times603\frac{123}{317}$?

Division.

In what does fractional division differ from multiplication by a fraction? — *Multiplication by a fraction values a part already known and numbered, by taking the value which appertains to it; division by a fraction numbers the parts already valued, singly, in the divisor.*

How is this demonstrated? — In fractional multiplication, the measure, or denomination, of the part to be taken is expressed by the terms themselves; as the $\frac{1}{2}$ of 6 dollars; the value of this is found, by the operation, to be 3 dollars.

How in division? — In division by a fraction, the quotient is always greater than the dividend; for if the dividend be integral or mixed, each unit of it will contain the fractional divisor more than once; if the dividend be an equal fraction, the quotient will have a unit; if the dividend be a smaller fraction than the divisor, the quotient will still be a larger fraction than the dividend.

Wherefore? — Because division is the reverse of multiplication; multiplication by a fraction reduces the other factor in every case, therefore division by a fraction must, in every case, increase the other factor; that is, in the result.

What is the inference from the whole? — Therefore, if, on division by a fraction, the quotient specified the value of the part numbered in the divisor, a part would be of greater value than the whole; for the whole is the dividend, and the dividend is less than the quotient; such a result would be absurd.

Then what does such a quotient express? — The quotient, on division by a fraction, expresses only the number of turns, or parts of a turn, which the divisor will go in the dividend; and these turns answer, as in every like case of an integral divisor, to the number of parts into which the dividend is to be severed.

Of fractions by integers.

Divide a fraction by an integer; what results? — A fraction divided by a whole number gives a still smaller fraction; as half a dollar, divided by 2, gives a quarter. $2 : \frac{1}{2} = \frac{1}{4}$.

What is the product of $\frac{1}{2}$ a dollar by $\frac{1}{2}$? — Half a dollar, multiplied into $\frac{1}{2}$, produces a quarter also; for multiplication by a fraction whose numerator is 1, is division by the denominator.

How then do you divide a fraction by a whole number? — To divide a fraction by a whole number, multiply it into the denominator.

Suppose you divide the numerator? — To divide the numerator, or to multiply the denominator, by the same factor, gives the same result.

When would you divide the numerator? — I would divide the numerator whenever it could be done without remainder.

Hence what rule? — *To divide a fraction by a whole number, multiply it into the denominator, or by it divide the numerator.*

Of integers by fractions.

What must be the quotient of a fraction in a whole number? — The quotient of a fraction in a whole number is greater than the dividend; for every unit contains the fractional divisor more than once.

How shall we obtain such a quotient? — If the denominator of the divisor be multiplied into the integral dividend, and their product be divided by the numerator, the result will be a quotient larger than the dividend.

How is this demonstrated? — The denominator of a proper fraction is always larger than the numerator; consequently, multiplication of the dividend into the denominator will increase it more than division by the numerator will diminish it.

Can you exemplify the case? — Divide 6 dollars into portions of $\frac{3}{4}$ of a dollar each, the quotient will be 8, a greater number than 6, the dividend; for six 4s are 24; and 3s in 24, eight. $\frac{3}{4} : 6 = \frac{24}{3} = 8$.

Eight what? — Eight portions, or turns of the divisor in the dividend.

Then what part of 6 dollars are $\frac{3}{4}$? — Three quarters of a dollar are $\frac{1}{8}$ of 6.

How do you learn that? — From inference; for the number of parts, valued each, in the divisor, at $\frac{3}{4}$, being 8, every single part must be $\frac{1}{8}$.

How does this process differ from multiplication? — It seems to be multiplication inverted; for thus far the results given by a fractional divisor have been increase by the denominator, and diminution by the numerator.

What is it lessens a fraction? — The larger the denominator, the smaller the fraction.

What effect will this have in division? — The larger the denominator of a fractional divisor, the smaller is the divisor itself, and the greater must be the quotient; but the quotient can only be made greater by multiplying the denominator of the divisor into the integral dividend.

Can you apply this to the numerator? — The larger the numerator of a fractional divisor, the greater is the divisor itself, and the less must be the quotient; but the quotient can only be lessened by dividing an integral dividend by the numerator of the divisor.

Hence the rule? — *To divide a whole number by a fraction, multiply it into the denominator, and divide their product by the numerator.*

Of fractions by fractions.

How shall we extend the principles last demonstrated to the case of fractional dividends? — Since the quotient by any fractional divisor is larger than the dividend, the excess can be obtained only, when the dividend is a fraction, by multiplying the denominator of the divisor into the numerator of the dividend, and the numerator of the divisor into the denominator of the dividend.

Can you demonstrate this? — The denominator of the divisor is larger than its numerator; therefore, if multiplied into the numerator of the dividend, it will increase the quotient more than the multiplication of its numerator into the denominator of the dividend will diminish the quotient.

What are the respective values of a tenth and a fifth of a dollar? — A tenth of a dollar is, in value, 10 cents; a fifth of a dollar is 20 cents.

Then what part of $\frac{3}{5}$ of a dollar are $\frac{3}{10}$? — Three tenths of a dollar are one half of three fifths; for $\frac{3}{10}$ are 30 cents, and $\frac{3}{5}$ are 60 cents.

Now can you show arithmetically how often $\frac{3}{10}$ are contained in $\frac{3}{5}$? — If the question be, what part of $\frac{3}{5}$ are $\frac{3}{10}$, three tenths are a divisor; ten 3s are thirty, the numerator of the quotient; three 5s are 15, the denominator; 15s in 30, two; hence $\frac{3}{10}$ of a dollar appear to be equal to $\frac{1}{2}$ of $\frac{3}{5}$, for it is one part out of two parts. $\frac{3}{10} : \frac{3}{5} = \frac{30}{15} = 2$.

How would you compare this with multiplication? — This is still fractional multiplication inverted; for, instead of multiplying similar terms together, we have multiplied the denominator of the divisor into the numerator of the dividend, and the numerator of the divisor into the denominator of the dividend.

Suppose the terms of the divisor to measure those of the dividend, what will be the course? — If a divisor, in both its terms, measure the corresponding terms of a dividend, divide numerator by numerator, and denominator by denominator.

How do you justify this mode of division? — Division is the reverse of multiplication, and in multiplication of fractions, we multiply corresponding terms one into the other.

Can you imagine no independent reason? — In dividing by a fraction, the denominator of the divisor is the term of increase, for the larger the denominator, the larger is the quotient; therefore we divide one denominator by the other, to enlarge the quotient: the numerator of the divisor is the term of diminution; for the larger the numerator, the smaller is the quotient; therefore we divide one numerator by the other to lessen the quotient.

Can you now recite the rule?

RULE OF FRACTIONAL DIVISION.

To divide a fraction by a whole number, multiply it into the denominator; or, if it measure the numerator, divide the numerator by it.

To divide a whole number by a fraction, multiply it into the denominator, and divide their product by the numerator.

To divide a fraction by a fraction, suppose the divisor inverted, and proceed as in fractional multiplication; the resulting fraction is the quotient: or if the divisor, in both its terms, measure the corresponding terms of the dividend, divide numerator by numerator, and denominator by denominator.

To divide by a mixed number, simplify the divisor, multiplying dividend or quotient into the same denominator.

N. B. The fraction of a mixed number may sometimes easily be converted into a decimal, as will be shown hereafter; thus the process of simplification may be avoided.

Proof. Fractional division is proved by fractional multiplication.

APPLICATION.

1. Divide $\frac{54}{70}$ by 7; what is the quotient?

$\frac{54}{70}=\frac{27}{35}$. $7:\frac{27}{35}=\frac{27}{245}$, answer.

The terms of the fraction being even, are divisible by 2; and the divisor, not measuring the numerator, is multiplied into the denominator. The terms have the same meaning, and the operation is precisely the same, as if the question had been one of multiplication, expressed in this manner: How much is $\frac{1}{7}$ of $\frac{54}{70}$? for, as so often repeated, multiplication by a fraction whose numerator is 1, is division by the denominator.

2. What part of 19 is $\frac{1}{9}$?

$\frac{1}{9}:19=171$; therefore $\frac{1}{9}=\frac{1}{171}$ of 19, answer.

$19\times\frac{1}{171}=\frac{19}{171}=\frac{1}{9}$, proof.

3. Divide $\frac{48}{49}$ by 6; what is the quotient? $6:\frac{48}{49}=\frac{8}{49}$.
Such easy examples need no formal proof.

4. How many times are $\frac{5}{7}$ contained in 63?

$\frac{5}{7}:63=\frac{441}{5}=88\frac{1}{5}$, answer. $88\frac{1}{5}\times\frac{5}{7}$

$7:441=63$, proof.

To have asked how much are $\frac{5}{7}$ of 63, would have been a very different question, and have required a very different solution.

5. What part of $\frac{7}{11}$ are $\frac{3}{8}$?

$\frac{3}{8}:\frac{7}{11}=\frac{56}{33}=1\frac{23}{33}$, answer. $1\frac{23}{33}\times\frac{3}{8}=\frac{3\frac{23}{11}}{8}=\frac{56}{88}=\frac{7}{11}$, proof.

A simplification of the product, obtained through the numerator of the multiplier, brings us readily to the proof.

6. What is the quotient of $43\frac{6}{7}$ in $17856\frac{5}{13}$?

$$
\begin{array}{lll}
43\frac{6}{7} & 17856\frac{5}{13} & \\
\ \ 7 & \qquad\quad 7 & \\
\hline
307: & 124994\frac{9}{13}=407\frac{45\frac{9}{13}}{307}\text{, ans.} & \\
 & 1228 & \\
\hline
 & \ \ 2194\frac{9}{13} & \\
 & \ \ 2149 & \\
\hline
 & \quad 45\frac{9}{13} &
\end{array}
\qquad
\begin{array}{l}
407\frac{45\frac{9}{13}}{307} \\
307 \\
\hline
\quad 45\frac{9}{13} \\
\ \ 2849 \\
1221 \\
\hline
124994\frac{9}{13}\text{, proof of division.}
\end{array}
$$

$$307 : 17856\tfrac{5}{13} = 58\frac{50\frac{5}{13}}{307} \qquad \frac{50\frac{5}{13}}{307} \times 7 = \frac{352\frac{9}{13}}{307} = 1\frac{45\frac{9}{13}}{307}.$$

$$\begin{array}{r} 58\frac{50\frac{5}{13}}{307} \\ 7 \\ \hline 407\frac{45\frac{9}{13}}{307} \end{array}$$, answer by multiplying the quotient.

The latter mode of obtaining the answer is attended with more complicated fractional multiplication than the former; as may be seen in the side work.

Examples to be wrought and recited.

What are the values, on the division of

1. $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, $\frac{1}{6}$, $\frac{1}{7}$, $\frac{1}{8}$, $\frac{1}{9}$, $\frac{1}{10}$, $\frac{1}{11}$, $\frac{1}{12}$, $\frac{1}{13}$, $\frac{1}{14}$, $\frac{1}{15}$, $\frac{1}{16}$, $\frac{1}{17}$, $\frac{1}{18}$, $\frac{1}{19}$, $\frac{1}{20}$, by 2?

2. $\frac{2}{3}$, $\frac{3}{4}$, $\frac{4}{5}$, $\frac{5}{6}$, $\frac{6}{7}$, $\frac{7}{8}$, $\frac{8}{9}$, $\frac{9}{10}$, $\frac{10}{11}$, $\frac{11}{12}$, $\frac{12}{13}$, $\frac{13}{14}$, $\frac{14}{15}$, $\frac{15}{16}$, $\frac{16}{17}$, $\frac{17}{18}$, $\frac{18}{19}$, $\frac{19}{20}$, by 3?

3. $\frac{3}{4}$, $\frac{5}{6}$, $\frac{6}{7}$, $\frac{7}{8}$, $\frac{8}{9}$, $\frac{9}{10}$, $\frac{10}{11}$, $\frac{11}{12}$, $\frac{12}{13}$, $\frac{13}{14}$, $\frac{14}{15}$, $\frac{15}{16}$, $\frac{16}{17}$, $\frac{17}{18}$, $\frac{18}{19}$, $\frac{19}{20}$, by 4, 5, 6, 7, 8, 9, successively.

4. $\frac{4}{5}$, $\frac{5}{6}$, $\frac{6}{7}$, $\frac{7}{8}$, $\frac{8}{9}$, $\frac{9}{10}$, $\frac{10}{11}$, $\frac{11}{12}$, $\frac{12}{13}$, $\frac{13}{14}$, $\frac{14}{15}$, $\frac{15}{16}$, $\frac{16}{17}$, $\frac{17}{18}$, $\frac{18}{19}$, $\frac{19}{20}$, by 10, 11, 12, 13, successively.

What are the parts, on the division of

5. 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, by $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, $\frac{1}{6}$, $\frac{1}{7}$, $\frac{1}{8}$, $\frac{1}{9}$, $\frac{1}{10}$, $\frac{1}{11}$, $\frac{1}{12}$, $\frac{1}{13}$, successively.

6. 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, by $\frac{2}{3}$, $\frac{3}{4}$, $\frac{4}{5}$, $\frac{5}{6}$, $\frac{6}{7}$, $\frac{7}{8}$, $\frac{8}{9}$, $\frac{9}{10}$, $\frac{10}{11}$, $\frac{11}{12}$, $\frac{12}{13}$, successively.

What are the parts, or values, on the division of

7. $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, $\frac{1}{6}$, $\frac{1}{7}$, $\frac{1}{8}$, $\frac{1}{9}$, $\frac{1}{10}$, $\frac{1}{11}$, $\frac{1}{12}$, $\frac{1}{13}$, $\frac{1}{14}$, $\frac{1}{15}$, $\frac{1}{16}$, $\frac{1}{17}$, $\frac{1}{18}$, $\frac{1}{19}$, $\frac{1}{20}$, by $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, successively.

8. $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, $\frac{1}{6}$, $\frac{1}{7}$, $\frac{1}{8}$, $\frac{1}{9}$, $\frac{1}{10}$, $\frac{1}{11}$, $\frac{1}{12}$, $\frac{1}{13}$, $\frac{1}{14}$, $\frac{1}{15}$, $\frac{1}{16}$, $\frac{1}{17}$, $\frac{1}{18}$, $\frac{1}{19}$, $\frac{1}{20}$, by $\frac{1}{6}$, $\frac{1}{7}$, $\frac{1}{8}$, $\frac{1}{9}$, $\frac{1}{10}$, $\frac{1}{11}$, $\frac{1}{12}$, $\frac{1}{13}$, successively.

9. $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, $\frac{1}{6}$, $\frac{1}{7}$, $\frac{1}{8}$, $\frac{1}{9}$, $\frac{1}{10}$, $\frac{1}{11}$, $\frac{1}{12}$, $\frac{1}{13}$, by $\frac{2}{3}$, $\frac{3}{4}$, $\frac{4}{5}$, $\frac{5}{6}$, $\frac{6}{7}$, $\frac{7}{8}$, $\frac{8}{9}$, $\frac{9}{10}$, $\frac{10}{11}$, $\frac{11}{12}$, $\frac{12}{13}$, successively.

10. $\frac{2}{3}$, $\frac{3}{4}$, $\frac{4}{5}$, $\frac{5}{6}$, $\frac{6}{7}$, $\frac{7}{8}$, $\frac{8}{9}$, $\frac{9}{10}$, $\frac{10}{11}$, $\frac{11}{12}$, $\frac{12}{13}$, by $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, $\frac{1}{6}$, $\frac{1}{7}$, $\frac{1}{8}$, $\frac{1}{9}$, $\frac{1}{10}$, $\frac{1}{11}$, $\frac{1}{12}$, $\frac{1}{13}$, successively.

The preceding examples are for mental solution also, on interrogation.

What are the values, on the division of

11. $\frac{7}{19}$ by 2? $\frac{10}{21}$ by 3? $\frac{5}{16}$ by 4? $\frac{6}{32}$ by 5? $\frac{8}{54}$ by 6? $\frac{3}{17}$ by 7? $\frac{9}{16}$ by 8? $\frac{11}{55}$ by 9? $\frac{4}{104}$ by 10?

12. $\frac{9}{99}$ by 11? $\frac{5}{73}$ by 12? $\frac{4}{19}$ by 13? $\frac{6}{25}$ by 11? $\frac{8}{58}$ by 12? $\frac{7}{96}$ by 13? $\frac{3}{83}$ by 21? $\frac{9}{55}$ by 34? $\frac{10}{42}$ by 20?

What are the parts, on the division of

13. 1008 by $\frac{3}{4}$? 7041 by $\frac{4}{7}$? 2351 by $\frac{6}{11}$? 8768 by $\frac{7}{12}$? 10328 by $\frac{9}{13}$? 56104 by $\frac{12}{37}$?

14. 2860 by $\frac{18}{33}$? 4594 by $\frac{11}{13}$? 6872 by $\frac{43}{99}$? 8309 by $\frac{60}{83}$? 4726 by $\frac{15}{56}$? 7941 by $\frac{31}{51}$?

What are the contents of

15. $25\frac{3}{4} : 304\frac{1}{2}$? $13\frac{3}{11} : 176\frac{5}{6}$? $166\frac{7}{35} : 929\frac{5}{13}$? $25\frac{9}{10} : 111$? $100\frac{17}{27} : 671\frac{23}{34}$?

16. $73\frac{19}{37} : 203\frac{98}{176}$? $98\frac{7}{16} : 555\frac{31}{41}$? $29\frac{4}{7} : 202\frac{30}{100}$? $78\frac{17}{19} : 7041\frac{21}{22}$? $103\frac{5}{17} : 4038\frac{55}{63}$?

DECIMALS.

Notation.

What are decimals? — Decimals are fractions whose denominator is 10 or its powers.

In what proportion does the numeration table increase to the left? — The numeration table increases to the left in a tenfold proportion.

What is its gradation to the right? — To the right as far as units, it diminishes in the same proportion.

If extended below units, what would be its scale of decrease? — A numeration table carried below units may continue to decrease in a tenfold proportion.

What would be the value of the nearest places? — The places nearest to units would be tenths, hundredths, thousandths, and so on.

Can you exemplify this in things? — The nearest places to the dollar unit are dimes, cents, and mills, in the order mentioned.

What is a myriad? — A myriad is ten thousand.

Can you now enumerate, from millions on the left to millions on the right?

NUMERATION TABLE WITH DECIMAL PARTS.

Millions.	Hund. of Thous.	Tens of Thous.	Thousands.	Hundreds.	Tens.
2000000	200000	20000	2000	200	20

Units.	Tenths.	Hundredths.	Thousandths.	Myriadths.	Hund. Thousandths.	Millionths.
2	·2	·02	·002	·0002	·00002	·000002

What is the order of succession in this table? — From units toward the left, the numeration table increases at every place, by multiplication into 10; toward the right, it diminishes at every place, by division by 10.

In the table, how are the decimal parts distinguished? — Tenths are distinguished by a point (.) before a single place on its right; hundredths, by a point before two places on the right, &c.

This point has hitherto been called the separatrix; what would seem a more expressive term? — From its property of decimating a number, perhaps we may be allowed to call it a decimator; especially as the comma, used for cutting off integral places, is appropriately named from its office of separating.

Should there be no decimals of the first place, how must the second be noted? — On the principle of an equal and constant diminution in the value of places toward the right, every place, as well below units as above, intervening between the due place of significants, must be filled with a cipher.

In what does the decimal differ from the integral notation in this respect? — The notation of decimals often requires ciphers on the extreme left; for the place of units is on their left.

Why should this constitute a reason? — Decimals, like integers, are valued by their distance from units; and this can only be known by every intervening place being filled either with ciphers or significants.

What do you say of ciphers on the extreme right? — Ciphers on the extreme right of decimals make no change in their value; for they indicate only a farther severance of the thing numbered.

The case of repetends is an apparent exception.

Can you instance this in things? — A dime is neither increased or diminished in value by a severance into 10 cents.

What then does every decimal place indicate? — Every successive place of decimals indicates an additional power of 10 in the denominator.

What is the inference? — *The denominator of a decimal contains as many ciphers after a unit as there are decimal places in the number expressed.*

Wherefore ciphers? — Because powers of 10 are expressed by a cipher after a unit for every power.

Decimal terminations.

How are decimals formed? — Decimals arise on the continuance of a division below units.

What limit may there be to such a division? — So long as there is a remainder, we may continue to divide it, by severing it into tenths of what it was before; or supposing it so severed.

Arithmetically, how would you sever it? — We make the severance, by annexing a cipher, or supposing it annexed; for this, while it leaves the value unchanged, increases the number of parts tenfold, and often enables us to take such parts in reality.

Can you instance it? — A dollar is severed into 10 parts, and made divisible into dimes, by annexing a cipher after a point; a dime is severed into 10 parts, and made divisible into cents, by annexing an additional cipher.

10 : \$1·0 = ·1, i. e. one dime. 10 : \$·10 = ·01, or one cent.

In this case, what may the termination be called? — When a divisor measures the dividend, the termination is definite, or perfect.

Can you decimally divide 1 by 9? — The division of 1 by 9 never comes to an end, for by calling the dividend, and every remainder, tenths, the quotient continually repeats the figure 1. 9 : 1 = ·111, &c.

What would you say of such a quotient? — A quotient that has no end may properly be called interminate; or, if spoken of as terminated at will, the termination may be called indefinite, for it has no definable limit of its own.

Why not infinite? — That which is infinite has no beginning, that which is indefinite in nature has no necessary ending.

Why do you add necessary? — Because we end such a division whenever we please, and resume it when we please.

Then what is meant by speaking of any division as infinite? — Infinite divisibility can mean only indefinite divisibility; for, if any thing more be intended by it, infinity being without commencement, as well as without end, infinite divisibility would seem to be infinite nonsense.

Repetends.

What common fraction would express the division of 1 by 9? — The common fraction representing the division of 1 by 9 is $\frac{1}{9}$.

What is every fraction? — Every fraction is a quotient.

What is the inference, as applied to the present case? — Therefore the quotient lately found ·111, &c. is equal to $\frac{1}{9}$.

Suppose you had stopped at the first digit? — Whatever number of figures of 1 may thus be repeated in a quotient, it is evident that $\frac{1}{9}$ would represent their true value; for it expresses absolutely and completely the part to be taken.

How could it express the true value of a quotient that is never complete? — The common fraction in reality expresses more than the decimal interminate ever could express; it is therefore but a substitute for the decimal.

What arithmetical reason authorizes it? — No arithmetical reason authorizes the substitution, that I can imagine, except the knowledge of the certain origin from which the uncertain decimal springs.

Can you divide $\frac{1}{7}$ in the same manner? — On dividing 1 by 7, six digits are perpetually repeated.

$$7 : 1 = \cdot\dot{1}4285\dot{7}\dot{1}4285\dot{7}, \text{\&c.}$$

What name may be given to digits which thus repeat? — *Decimal places that repeat the same figure or the same series of figures, at certain intervals, are called repetends and circulates.*

How would you distinguish them from other decimals? — Repetends are distinguished by a point over the first and last places of their circle, or over any singly repeating figure.

What must be the limit of any repeating circle? — The circle of a repetend will continue till a former remainder repeat, and no longer; for in a continued decimal division, remainders form new dividends with the annexation of ciphers; and like factors produce a like quotient.

What may be the greatest number of repeating places? — The greatest number possible of repeating places is equal to the whole number of units in the divisor, less one.

How is this demonstrated? — The figures of a quotient vary as the dividend varies; but when new dividends are formed by the annexation of ciphers only to remainders, there can be only as many different dividends as there are units in a divisor, less one; for one additional must proceed from a remainder equal to the divisor; but a number equal to the divisor cannot be a remainder.

When is it that new dividends are thus formed? — After exhaustion of the significants in a dividend that does not itself repeat, we can extend it by annexing ciphers only; repetends then begin to appear, for the dividends thereafter can only vary as the remainders vary.

What decimal quotients repeat? — All decimal quotients, not exact measures, continue themselves by repeating, if, on the exhaustion of the dividends, they can be extended only by ciphers; but, unless the divisor be small, the repeating circle may be so large as to be useless.

Have you an instance of the limitation of places? — On the division of 1 by 7, the repeating places are six.

How far would you continue to divide in these cases? — Repetends of a moderate compass should be continued to the end of their circle, and no farther; for we then know what figures succeed.

Divide any number of digits of 1 by as many 9s; what is the quotient? — On dividing 111 by 999, I find the dividend continually repeated.

What is the value of the quotient? — The quotient may be valued by the other terms of the division, as $\frac{111}{999}$; and this may be reduced to $\frac{1}{9}$, for the numerator is a measure of the denominator.

$$
\begin{array}{r}
999 : 111{\cdot}0 = {\cdot}11\dot{1} \\
999 \\
\hline
1110 \\
999 \\
\hline
1110 \\
999 \\
\hline
111
\end{array}
$$

What is the decimal quotient of $\frac{3}{9}$? — Three divided by 9, continually repeats the same figure of 3.

$$9 : 3 = {\cdot}33\dot{3}, \text{ \&c.} = \tfrac{333}{999}.$$

How would you value this repetend? — The repetend $\cdot 33\dot{3}$, &c. is equal to a common fraction, having as many 9s in the denominator as it has 3s in the numerator.

Why? — The fraction $\frac{3}{9}$ has a dividend three times larger than the fraction $\frac{1}{9}$; its quotient therefore must be three times larger.

Divide the three first digits by as many 9s; what will be the quotient? — The quotient derived from the division of 123 by 999, constantly repeats the dividend, as in a circle.

To what is this owing? — Any number of 9s, short of an exact measure, can be contained in an equal number of places, plus one, filled with a cipher, only as many times complete as are expressed in the left hand figure.

$$
\begin{array}{r}
999 : 123{\cdot}0 = {\cdot}\dot{1}2\dot{3} \\
999 \\
\hline
2310 \\
1998 \\
\hline
3120 \\
2997 \\
\hline
123
\end{array}
$$

How is this demonstrated? — Such a dividend will always contain the divisor once for every unit of its left hand digit, and something more than once; for the left hand *place* of the dividend always exceeds in value the whole divisor, at the least, by a unit; as 10 exceeds 9; as 100 exceeds 99, &c.

Will the something more never amount to an additional unit in the quotient? — Should every place of the dividend be filled with the largest digits possible under the maxim, as with the number, 890; still, the excess from the highest place will be a unit short of increasing the lower places to an equality with the divisor.

$$99 : 800 + 90 = 8, \text{ \&c.} \qquad\qquad 99 : 890 = 8\tfrac{98}{99}$$

792	792
8˙	98

What is the inference? — The first quotient therefore arising from such factors must, in every case, repeat the left hand figure of the dividend.

What of the succeeding places? — Every remainder on such a division must consist of the next succeeding figures of the dividend, terminated by the figure preceding.

How is this demonstrated? — The product of the divisor, by the quotient, is equal to as many tens, hundreds, &c. as are expressed by the left hand place of the dividend, less as many units; the dividend therefore ending in a cipher, its left hand digit must re-appear in the units remaining, for there is nothing to increase them; and further, the product of the divisor falling short of the left hand place of the dividend, those on the right, with the exception of the cipher on the extreme right, already considered, must also re-appear in the remainder.

What may be inferred from this? — The dividend therefore, prolonged by ciphers, must be in continual rotation; and as its left hand figure comes anew to the left in successive remainders, must renew its appearance in the quotient.

What is your conclusion from the whole? — *The fractional denominator of a repetend consists of as many digits of* 9 *as the repetend contains decimal places.*

Is the conclusion universally applicable? — The conclusion is true, let the digits of a repetend be what they may; for, if the repeating figures of any quotient be made a dividend, and be divided by as many 9s as there are repeating places, the figures of that dividend will re-appear in the quotient and be of the same value as in the first quotient; the quotients therefore being the same, the factors must either be the same, or be in the same proportion one to the other.

Partial repetends.

Can you divide a tenth into ninths decimally? — A tenth can *never be* measured by 9; but by severing it into hundredths,

&c., a decimal division may be prolonged; as 9 parts, nearly, may be taken of a dime, by changing it to cents.

Thus divided, what is the quotient given? — A tenth divided by 9 gives a hundredth to each part, and something over.

How must a hundredth part be expressed by the decimal notation? — Noted decimally, a hundredth part can only be expressed by a cipher prefixed to a unit, both on the right of the decimator; for hundredths are in the second place of decimals, and every place intervening from units must be noted.

$$9 : \cdot 1 = \cdot 011\dot{1}.$$

In the quotient, what figures repeat? — The repetend is the unit only, the cipher not repeating.

What may such quotients be called? — Decimals consisting of non-repeating and repeating digits may be called partial repetends.

What will be the notation? — The notation is by the repeater set over repeating figures only. $9 : \cdot 1 = \cdot 0\dot{1} = \frac{1}{90}$.

How is such a decimal valued? — *The fractional denominator of a partial repetend is a number of* 9*s equal to its places, multiplied into the power of* 10 *indicated by the places not repeating.*

Why is this? — Because decimal places diminish in value ten times for every place of their removal from units, and diminution is effected by multiplying a denominator.

Then what is the value of the quotient last obtained? — The fractional value of the quotient $\cdot 0\dot{1}$ is $\frac{1}{90}$; for the value of the repetend one, is $\frac{1}{9}$; but a repetend of the second place is of ten times less value than one of the first.

Can you illustrate this from things? — A ninth part of a dime ($\cdot\frac{1}{9}$) is equal to a ninetieth part ($\frac{1}{90}$) of a dollar; and if dollars were coined into ninetieths instead of hundredth parts, one such ninetieth would be an exact share on such a division.

What effect have ciphers that repeat on the right? — Repetend ciphers on the right lessen the value of decimal fractions, for they increase denominators, adding ciphers only to numerators.

Approximates.

Can you now explain why many decimal divisions never come to an end? — Indefinite terminations are owing to our attempting to perform contradictory things; we cannot divide a dime or a dollar into exactly nine parts, because there is not an exact number of 9s in 10, or 100.

How do you apply the observation to figures? — Decimal terminations seldom come to an end, because we cannot sever a number into tenths, hundredths, and so forth, to part them afterward into thirds, sevenths, ninths, elevenths, &c., the two things being, in their own nature, inconsistent one with the other.

Strictly speaking, is such a number ever divided? — A dividend that is not the multiple of some quotient never is, and never can be, divided, i. e. measured: it may be severed unequally; but the number divided is an approximate substitute; an exact multiple, namely, of the quotient, wherever terminated.

Why do we employ the decimal division? — The decimal division is used for the sake of a uniform, and therefore easy, mode of computation.

How does this facility appear? — By means of decimals, fractional operations are conducted precisely as integral; whereas common fractions, having denominators of every amount, usually require a double operation.

Are not the inconveniences of indefinite terminations obviated? — In practice it may be often impossible to divide a thing into tenths; but in accounts we readily approach to a perfect division within any needful degree of accuracy.

Can you exemplify this from the late instance? — If we continue the division of a dime by 9, to three places, we obtain a mill in addition to the cent; and although mills are not coined, yet, when we have to multiply, mills may be increased to cents.

What are fractions thus approaching to a true value called? — *Quotients interminate with respect to the dividend are called approximates.*

What is the use of repetends in this respect? — Repetends are occasionally of use in decimal operations, by enabling us to extend factors as far as we please with true figures; and thus to obtain a truer result; sometimes one that is exact.

What must be done when perfect accuracy is required, and decimals cannot be exactly measured? — Perfect accuracy may require the use of common in preference to decimal fractions, or of decimals made determinate by means of a common fraction on the right.

Can you exemplify this? — We might terminate the repetend $\cdot 1$ by $\frac{1}{9}$, for this also is its true value; since one tenth added to one ninth of another tenth, is equal to $\frac{10}{90} = \frac{1}{9}$.

$$\cdot 1\tfrac{1}{9} = \tfrac{1}{10} + \tfrac{1}{9 \times 10} = \tfrac{10}{90} = \tfrac{1}{9}.$$

When will it be useful thus to terminate a decimal? — It

can only be useful to terminate a decimal with a common fraction, when the denominator is tabular, or easily operated with.

What kind of decimals would you thus terminate? — Common fractions may usefully terminate decimals strictly called approximate.

Why do you add strictly? — Strictly, because repetends are approximate also, but easily valued with accuracy; other approximates never can be correctly valued in practice, their circle being too large to be known.

Can you instance this? — A divisor of only two places may give a repetend of ninety-eight places; such quotients therefore are strictly called approximate.

How far may it be useful to carry approximates? — Approximates should always be carried so far as to supply a true figure for every turn we mean to make in division; and to render the lowest computative value to products in multiplication; as of mills in money.

Why need they be carried no farther? — This limit is sufficiently extended in results; for beyond, and often within it, the parts deficient are nothing to us, being unsusceptible of discharge; as of mills fewer than the value of half a cent.

What should be the usual extent? — Approximates should seldom fall short of the 6th place, or millionths.

Can you now state the rule of decimal notation?

RULE OF DECIMAL NOTATION.

Decimals are figured like integers, but after a central point on the left; and from the left they are enumerated.

Repetends are noted by a point over the repeating figure; over the first and last decimal also of a circle that repeats.

Reduction.

How are decimals reduced to common fractions? — Decimals are reduced to common fractions by suffixing 10, or the power of 10 corresponding with their places, as denominators; or by reduction at once to their least terms.

Are approximates thus reducible? — Approximates may have their denominators suffixed, but are, in no just sense, reducible to common fractions of an immeasurable dividend; for a common fraction is a certain definite part, which an approximate never can be.

But if terminated by a common fraction? — A fractional termination renders them definite.

How are repetends reduced? — Repetends are reduced to the common form by suffixing digits of 9 as denominators, the same in number with their places.

Partial repetends how? — Partial repetends are complex fractions, having digits of 9 as denominator of part of their numerator, and powers of 10 for their general denominator; they are simplified therefore by multiplication of both terms into the complicating denominator.

A common to a decimal fraction? — Common fractions are necessarily reduced to decimals by carrying out the division. So $\frac{1}{5}$ becomes two tenths. $5 : 1 = \cdot 2.$

What divisors give exact decimal quotients? — The numbers, 2, 4, 5, 8, 10, and their powers, bring their quotients to an exact termination.

Owing to what? — Because 2, 5, 10, measure every single ten; 4 and 8 are powers of 2, and measure therefore some power or product of 10, necessarily to arise in a continued decimal division.

Do no other divisors effect the same thing? — All divisors commensurate with their dividend, submultiples of it, or measuring some product of it by a power or product of 10, bring their quotients to an exact termination.

Can you give an instance? — Six do not measure 3, but decimally they divide 3·0, the product of 3 multiplied into 1·0.

$6 : 3 = \cdot 5.$

Can you now bring these methods together?

RULE OF DECIMAL REDUCTION.

To reduce a common to a decimal fraction, divide numerator by denominator, annexing as many ciphers on the right of the numerator as may be required for a single division, and prefixing to the quotient, after a point, the same number of ciphers, less one. Continue to divide as in whole numbers, annexing a cipher or more to every remainder, till it repeat, or the quotient be sufficiently extended.

Tabular divisors are used with the same advantage as in integral division.

To reduce decimal to common fractions, omitting to prefix ciphers on the left of the numerator, suffix, for denominators, the power of 10 corresponding with all the decimal places, and reduce to the least terms.

To reduce repetends to common fractions, omitting repetend ciphers on the left, suffix as many digits of 9 as there may be repeating places in all.

To reduce partial repetends to common fractions, omitting

ciphers on the left, make other places the numerator of a complex fraction, having the power of 10 indicated by non-repeating places, for its general denominator; multiply both terms into as many digits of 9 as there may be repetends, and reduce to the least terms.

Approximates made definite by fractional terminations are reduced in the same manner, by using the denominator of their complex term for the simplifying process.

APPLICATION.

1. What is the decimal equivalent to the common fraction, $\frac{7}{80}$?

$8{,}0 : 7 = \cdot 0875$, answer. $\quad \cdot 0875 = \frac{875}{10000} = \frac{35}{400} = \frac{7}{80}$, proof.

According to the rule of division, a cipher on the right of a divisor is cut off; this requires a corresponding decimation of the dividend, at the outset, which, since a fraction can give no integer to the quotient, is best made at once, by prefixing the point; but as the divisor will not even then go once in the dividend, we know that the first place it can divide is that of hundredths, and accordingly fill the place of tenths in the quotient with a cipher. Having found the decimal, we prove it by a second reduction to the fractional form: in doing this, we employ 25, as a divisor easily operated with, and often measuring terms that end in 5 or a cipher.

2. What fractions are equivalent to the repetends $\cdot\dot{9}\dot{0}$ and $\cdot\dot{0}\dot{9}$?

$\cdot\dot{9}\dot{0} = \frac{90}{99} = \frac{10}{11}$; $\quad \cdot\dot{0}\dot{9} = \frac{9}{99} = \frac{1}{11}$; answer.

From the first example of these two, we may perceive what would be the mischief of omitting repetend ciphers on the right; it would there increase the fraction to a unit; for $\frac{9}{9} = 1$.

3. What is the fractional value of the partial repetend $\cdot 41\dot{6}$?

$\cdot 41\dot{6} = \frac{41\frac{6}{9}}{100} = \frac{375}{900} = \frac{5}{12}$, ans. $\qquad 12 : 5 = \cdot 41\dot{6}$, proof.

4. What is the fractional value of the partial repetend $\cdot 0\dot{6}$?

$\cdot 0\dot{6} = \frac{\frac{6}{9}}{10} = \frac{6}{90} = \frac{1}{15}$, ans. $\qquad 15 : 1 = \cdot 0\dot{6}$, proof.

The cipher on the left indicating only a diminution of the fraction to a tenth part of six ninths, it is obviously, without process, reducible to $\frac{6}{90}$.

5. What is the common fraction equivalent to the definite $\cdot 0089\frac{2}{7}$?

$\cdot 0089\frac{2}{7} = \frac{89\frac{2}{7}}{10000} = \frac{625}{70000} = \frac{25}{2800} = \frac{1}{112}$, answer.

$112 : 1\cdot 000 = 0089\frac{32}{112}$. $\quad \frac{32}{112} = \frac{16}{56} = \frac{2}{7}$, proof.

```
896
----
1040
1008
----
  32
```

Examples to be wrought and recited.

What decimals are equivalent to the common fractions

1. $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, $\frac{1}{6}$, $\frac{1}{8}$, $\frac{1}{9}$, $\frac{1}{11}$, $\frac{1}{12}$, $\frac{2}{3}$, $\frac{3}{4}$, $\frac{2}{5}$, $\frac{5}{6}$, $\frac{3}{8}$, $\frac{4}{9}$, $\frac{2}{11}$, $\frac{7}{12}$, $\frac{1}{20}$, $\frac{7}{30}$, $\frac{3}{40}$, $\frac{3}{50}$, $\frac{7}{60}$, $\frac{3}{80}$, $\frac{7}{90}$?

What are the fractional values of

2. $\cdot 5$, $\cdot 25$, $\cdot 75$, $\cdot 125$, $\cdot 05$, $\cdot 025$, $\cdot 075$, $\cdot\dot{1}$, $\cdot 0\dot{1}$, $\cdot\dot{0}0\dot{1}$, $\cdot\dot{3}$, $\cdot\dot{0}\dot{9}$, $\cdot\dot{9}\dot{0}$, $\cdot\dot{6}$?

The preceding examples are for mental solution also; the repetends and partial repetends being designed to be mentioned as such.

What are the decimal values of

3. $\frac{11}{13}$, $\frac{15}{45}$, $\frac{22}{63}$, $\frac{17}{38}$, $\frac{56}{108}$, $\frac{44}{95}$, $\frac{33}{74}$, $\frac{19}{29}$, $\frac{30}{31}$, $\frac{71}{94}$?

What are the fractional values of

4. $\cdot 459$, $\cdot 4448$, $\cdot 605$, $\cdot 7312$, $\cdot 6127$, $\cdot 80018$, $\cdot 01\dot{6}$, $\cdot 08\dot{3}$, $\cdot 41\dot{6}$, $\cdot 1\dot{6}$, $\cdot 008\dot{3}$, $\cdot 0208\dot{3}$?

5. $\cdot 041\dot{6}$, $\cdot 1174\frac{5}{7}$, $\cdot 701008\frac{7}{9}$, $\cdot 593410\frac{1}{6}$, $\cdot 623709\frac{4}{13}$, $\cdot 325173\frac{11}{12}$, $\cdot 356\frac{1}{24}$?

Addition.

What means the word addenda?—*Addenda* is a Latin word, signifying things to be added.

How would five cents be figured?—To figure five cents, the digit 5 is written after a point and a cipher; for cents are in the place of hundredths. 5 cents = $·05.

How are ten?—Ten cents are figured either as a dime, by a unit only after a point, or by a unit and cipher, to denote cents. 1 dime = $·1 = ·10.

How may five cents be added to other five, in figures?—By arranging the two decimals, representative of 5 cents each, one under the other, we may add them up, and obtain the sum total, which is 10 cents.

$$\begin{array}{r} \$\cdot 05 \\ \cdot 05 \\ \hline \cdot 10 \end{array}$$

Must the point be prefixed to the sum total?—Should the decimator be omitted, we could not distinguish between dollars and cents.

In adding mixed numbers, how are decimals to be placed?—Integers must be set under integers, and decimals under decimals; otherwise parts and whole numbers being mingled together, all would be confusion in the sum total.

What order must be followed?—As units are placed under units, tenths will necessarily fall under tenths, &c.

How would you arrange an approximate made definite by a common fraction? — Decimals fractionally terminated, and extending beyond the other addenda, may be allowed to retain their form; but should the common fraction intervene, its value must, as nearly as possible, be expressed decimally.

How may repetends be added? — Repetends mingled with other decimals, must be extended by their proper figures, and added in the manner of other decimals; it being impossible to make allowance for the slight difference of their denominators.

How when added to other repetends? — Repetends of a single figure are best added together, by dividing their sum total by 9 instead of 10, and carrying accordingly; pure conterminous circles are added together with sufficient accuracy for most purposes, by adding to the sum of their right hand column whatever tens would be carried from an adjacent column if figured.

Upon what principle? — Because we know what the figures of an adjacent column would consist of; and since it is commonly difficult and often impracticable, to attain perfect accuracy in operating with repetends, it may be well to make what approach to it we can, when unattended with perplexity.

When perfect accuracy is needed, what must be the course? — To obtain exact results from repetends, they must be reduced to the form of common fractions.

What is meant by conterminous repetends? — Conterminous repetends are such as correspond in circles or single figures, so as to begin and terminate together without the breach of a circle.

Can you now state the rule?

RULE OF DECIMAL ADDITION.

Decimal addenda are to be set, units under units, tenths under tenths; and to be added up as integers, from the right, having the decimator of the sum total placed in the same column exactly with the decimators above.

Repetends, pure or partial, mingled with other decimals, are to be extended by their proper figures, and added up as decimals; when added to repetends, their circles are to be made conterminous and their right-hand place increased by whatever number would be carried from an adjacent right hand column; or, if perfect accuracy be desired, they must be reduced to the form of common fractions.

Proof. The addition of decimals rightly placed is proved, like that of integers, by adding both upward and downward; but their due arrangement can only be judged of by inspection.

APPLICATION.

Addenda.	Addenda.
450·503	$\cdot\dot{9}\dot{0}$
29·91	$\cdot6\dot{6}$
3728·	$\cdot8\dot{8}$
210·0009$\frac{3}{8}$	$\cdot8\dot{8}$
·07	
4418·4839$\frac{3}{8}$, total.	3·34

$\frac{90}{99}+\frac{66}{99}+\frac{88}{99}+\frac{88}{99}=99:332=3\cdot\dot{3}\dot{5}$, perfect answer.

Examples to be wrought and recited.

What is the sum total of

1. 34·081+4·001+·0382+3+154·5+·76429+86·041+374·50671?
2. 90·1301+175·69342+56·2354+76·1+15·0055+23+·149959?
3. 7108·0001+326·724+·54101+·0006+7+1·910563+17·059?
4. ·0041+6823·2+4·095+138·76539+115·69+2736+73·0504+14?
5. 16·333+·55+·06713+8000+7401·0504+31756·79121+5831·074?

Subtraction.

How would you arrange decimals in subtraction? — As in addition, so in subtraction, units must be placed under units, tenths under tenths.

In what manner should approximates made definite be disposed? — Decimals terminated fractionally may be made terms in subtraction if their fraction extend beyond the term below or above; if not, their decimal places may be extended.

Wherefore? — There would otherwise be, in every such case, the inconvenience of subtracting from fractions of different denominators; in many, a misplacing also of parts.

Can you instance it? — From 25 cents we can easily subtract 6¼ cents, because a unit is reduced without labor to fourths; but if the common fraction were placed under cents, the cents of the subtractor would be set under the dimes of the minuend.

$·25
·06¼
·18¾, diff.

How are repetends operated with? — In subtraction, repetends are operated with as approximate, unless both terms be repetends and conterminous.

How in that case? — In subtracting repetend from repetend, should a unit be taken, the right hand figure of the difference is always one less, than if subtracted as a decimal;

because the unit is reduced to one part less, corresponding with the smaller denominator.

Can you exemplify this? — From the mixed repetend, $6\cdot\dot{6}$, take the repetend $\cdot\dot{9}\dot{0}$, the difference is not $5\cdot\dot{7}\dot{6}$, but $5\cdot\dot{7}\dot{5}$, because the unit taken is reduced only to 99 parts.

$$\begin{array}{r} 6\cdot6\dot{6} \\ \cdot\dot{9}\dot{0} \\ \hline 5\cdot\dot{7}\dot{5}, \text{ diff.} \end{array}$$

$1\frac{6}{99} = \frac{165}{99}$. $\frac{1\;5}{99} - \frac{90}{99} = \frac{75}{99}$.

How are repetends carried out? — *Repetends, in every case of every operation, are to be extended by their proper circles or single figures.*

Can you now state the rule?

RULE OF DECIMAL SUBTRACTION.

In decimal subtraction, set the subtractor under the minuend, units under units, tenths under tenths, &c. Subtract, as with integers, from the right, placing the decimator of the difference in the same column exactly with the decimators above.

Repetends are operated with as approximate, unless both terms repeat, and are conterminous; in which case, if a unit be taken, make the extreme right one less than it would otherwise be.

Approximates made definite can be used in that form only when the common fraction of one term extends beyond the other term.

Proof. Decimal subtraction is proved by decimal addition.

APPLICATION.

What remains from one dollar and a cent, less 29 mills?

$$\begin{array}{rl} \$1\cdot01 & \text{minuend.} \\ \cdot029 & \text{subtractor.} \\ \hline \cdot981 & \text{difference.} \end{array}$$

Examples to be wrought and recited.

What remains from

1. 23·054—16·7091? 561—9·034216? 77·7608—76·9504567?
2. 383·17—209·37856? 6·0005—·0324109? 11909—9988 999?
3. 1—·999999? 3600·009—3599·9999? 984·7—100·0001?
4. 99999—10000·9999? 7368·05—602·035? 15691·768—14000?
5. 43210·0076—38·764? 90001·90563—9000·97321$\frac{1}{7}$?

Multiplication.

How should decimal factors be arranged in multiplication? — Numbers that increase and diminish in the same proportion have the same properties; therefore decimal may be arranged like integral factors.

How shall the fractional results be distinguished? — *Decimal factors in multiplication produce as many decimal places as they themselves contain;* their number therefore may be noted in a product.

Can you demonstrate this? — Fractions, multiplied into whole numbers, divide the product by their denominators; multiplied into fractions, they multiply their denominators, one into another, for a divisor; thereby to divide the product of their numerators; but division by 10 and its powers is made, by pointing off as many places on the right of a product as the divisor contains powers of 10.

May not products exhibit fewer decimals than are to be found in the factors? — A product showing fewer places than the factors contain decimals, can arise only from purely fractional factors; which, as they diminish each other, may not produce figures of value to fill places answering to the product of their denominators.

Can you exemplify the case? — Let ·3 be multiplied into ·3; the significant produced is 9; but this 9 is of the value only of hundredths; for it is $\frac{3}{10}$ of $\frac{3}{10}$; to denote which, a cipher must be prefixed after the point. $\cdot 3 \times \cdot 3 = \cdot 09$.

What is the general inference? — *When a product exhibits fewer places than there are decimals in the factors, ciphers must be prefixed to equalize them.*

Why not annex ciphers? — Ciphers on the right of significant decimals do not show their distance from units; can therefore make no change in their value.

Can approximates made definite be used in multiplication? — Decimals with fractional terminations may be used precisely as mixed factors; for the farthest significants of both being placed under each other, any common fraction will be on the right of both.

Will the terms of such a fraction derive any decimal places from the other factor? — A common terminating a decimal fraction must not be noted with any decimal places as from another factor; for the value of decimals is noted from units exclusively, and a common terminating fraction is part only of the last decimal place.

How may repetends be used? — Repetends are of some use

in multiplicands; for since we know what would be the figure adjacent to the farthest right, if expressed, we can carry any number that would arise therefrom to the first partial product; if the repetend be single, we can divide its product, or the first partial product, by 9, instead of 10; and the entire product will be exact.

Have you an instance? — The pound sterling is usually valued at four dollars, forty-four cents, and four mills; that is, at 4 units and a repetend of 4; dividing the first partial product by 9, we obtain the precise amount of sterling money expressed by the multiplier.

$$\begin{array}{r} \$4{\cdot}44\dot{4} \\ 9£ \\ \hline \$40{\cdot}000 \end{array}$$

or,

$$\begin{array}{r} \$4{\cdot}\dot{4} \\ 9£ \\ \hline \$40{\cdot}0 \end{array}$$

What prevents the use of repetend factors universally, as such? — Except in the case of a single repeating figure, and that to be used in the multiplicand, it is impossible to obtain an accurate product from repetends without reducing them to common fractions; the reason of which is, that the decimal notation of denominators cannot truly represent a product arising from mixed decimal and common fractions.

What then must be done with repetend factors? — Repetend factors must be extended as approximate.

Can you now state the rule?

RULE OF DECIMAL MULTIPLICATION.

Arrange and multiply decimals like integers, pointing off as many places in the product as there are decimals in both factors; or prefixing as many ciphers as will equalize its decimal places to theirs.

Reduce repetends to common fractions; or extend them as approximates, and to every first partial product of a repeating multiplicand, carry the tens that would arise from an adjacent place, if expressed; or if the repetend of a multiplicand be single, divide its first partial product by 9 instead of 10.

N. B. Let the tens be carried in the case of repetends after the manner to be described under the head of contracted multiplication.

Proof. Decimal multiplication is proved by decimal division.

This being the case, we reserve the application till we can unite the processes under both rules; multiplication however alone is so simple, that immediate reference may be had to examples.

Quotients.

How should decimal factors be arranged in division? — Numbers that increase and diminish in the same proportion have the same properties; therefore decimal may be arranged like integral factors, both in multiplication and division.

When a decimal is divided by a whole number, what will be the nature of the quotient? — An integral divisor of a decimal can give no integral place to the quotient, for it is not once contained in the dividend.

When divided by a mixed number? — Still less can a mixed number; for the fraction annexed to a divisor increases it.

When divided by a decimal? — One decimal dividing another may give integral places to the quotient; for the divisor may be contained in the dividend once, or oftener.

What will such integers represent? — Integers in a quotient given by any fractional divisor represent turns only.

Divide a whole number by a decimal, what will the quotient be? — A decimal cannot divide an integer without having at least one integral place in the quotient; for any fraction is less than a unit, is therefore contained within it.

Divide an integer by an integral or mixed number, what places may the quotient have? — Any divisor that does not measure the dividend will give decimals to the quotient, by a continued division of the remainder.

How are remainders converted into new dividends? — By severing remainders into tenths, an effect produced by the annexation of ciphers or of repetends, we increase the number of the parts, without changing the entire value.

How is the effect produced by a repetend? — The annexation of a repetend produces more than the decuple of the number preceding; since it not only adds a place, but certain parts therewith.

Should a remainder require two places additional what is done? — The annexation of more places than one to a remainder is noted as in the division of whole numbers, by a cipher given to the quotient for every such additional place.

How is one fraction divided by another? — The usual mode of dividing fraction by fraction is to suppose the divisor inverted, and to proceed as in multiplication.

Is there no other method? — Division being the reverse of multiplication, we may divide numerator by numerator, and denominator by denominator.

Can you give me an example? — Divide $\frac{9}{28}$ by $\frac{3}{4}$; 3s in 9

three; 4s in 28, seven; the quotient is $\frac{3}{7}$; invert the divisor, and multiply; the quotient is $\frac{36}{84} = \frac{3}{7}$.

$$\tfrac{3}{4} : \tfrac{9}{28} = \tfrac{3}{7}. \quad \tfrac{3}{4} : \tfrac{9}{28} = \tfrac{36}{84} = \tfrac{3}{7}.$$

When is this method useful? — This is the neatest method of dividing a fraction, when the corresponding terms are multiples and measures.

Which method is employed in decimal operations? — This mode is constantly used in the division of decimals; for the numerators are the only apparent factors.

Do decimal divisors always measure their dividends? — Decimal denominators are always multiples or submultiples one of another; and for the quotients of numerators we content ourselves with approximation, if the factors give not a measure.

What then will be the number of decimal places in a quotient? — *The decimal places of a quotient will be as many as those of the divisor are fewer than those of the dividend.*

How is this demonstrated? — One decimal denominator either measures another on division, giving a unit only to the quotient, which is equivalent to an obliteration of both; or it gives that power of 10 which itself needs to equal the dividend; but this power is noted by pointing off an equal number of places in the quotient of the numerator.

Can you exemplify this? — Decimals may be noted as common fractions; therefore divide $\frac{24}{100}$ by $\frac{3}{10}$, and the quotient will be seen to have that power of 10, which the denominator of the divisor wants to bring it up to the denominator of the dividend. $\frac{3}{10} : \frac{24}{100} = \frac{8}{10}$. Decimally, $\cdot 3 : \cdot 24 = \cdot 8$.

When the divisor has no decimals, how will the quotient appear? — Without decimals in the divisor, the quotient must, of course, have one for every decimal place in the dividend, there being no other denominator to lessen them.

Should there be none in the dividend? — Without decimals in the dividend, there can be none in the quotient; for there is no denominator to be divided.

But what do you say of annexations? — Annexation to remainders is the same in effect as if made to their dividends; for remainders are parts of a dividend.

What integral places then will decimal divisors give to a quotient? — *A pure or mixed decimal divisor gives integral places to the quotient, till the decimals of the dividend exceed those of the divisor.*

How does this appear? — This follows necessarily from what has been shown concerning the number of decimal places in a quotient.

Can it not be shown independently? — If a dividend have no denominator, the denominator of a divisor is a multiplier of it; for the larger the denominator, the less is the divisor, and the larger the quotient; and if the numerator of the divisor exactly measure the dividend, there can then be no fractional place.

How in the remaining cases? — If a dividend have a denominator less than that of the divisor, its numerator will be more increased than diminished; and if the product of the numerator of the divisor, multiplied into the denominator of the dividend, measure its numerator thus increased, there can be no fractional place.

How in the case of equality? — If the denominators of divisor and dividend be equal, they measure each other, and the quotient is unity.

Can you exemplify these cases? — Divide 24 by $\frac{3}{100}$, the quotient is 800; divide $\frac{24}{10}$ by $\frac{3}{100}$, the quotient is 80; divide $\frac{24}{100}$ by $\frac{3}{100}$, the quotient is 8; all without decimal places.

$$\begin{cases} \frac{3}{100} : 24 = 8 \times 100. & \frac{3}{100} : \frac{24}{10} = 8 \times 10. & \frac{3}{100} : \frac{24}{100} = 8; \text{ or} \\ \cdot 03 : 24 = 800. & \cdot 03 : 2 \cdot 4 = 80. & \cdot 03 : \cdot 24 = 8. \end{cases}$$

What is the inference? — *When a quotient exhibits fewer digits than there are decimal places in the divisor, ciphers must be annexed on the right of the quotient to equalize them.*

Can you illustrate this by any special example? — Divide the dollar unit by five hundredths, and the quotient is 20; for there are 20 portions of 5 cents each in one dollar.

$$\cdot 05 : 1 = 20.$$

Suppose a different state of things, that as many decimal places do not appear in a quotient as the rule requires; what must be done? — *When the digits of a quotient are fewer than the excess of the decimal places of the dividend above those of the divisor, ciphers must be prefixed to equalize them.*

Can you exemplify this case? — Divide five thousandths by five tenths, the quotient is one hundredth. $\cdot 5 : \cdot 005 = \cdot 01$.

Separation of Ciphers, &c.

When ciphers are cut off from a divisor, what is the proceeding? — *Ciphers cut off from a divisor require the separation of integers from the dividend; or a quotient containing more decimal places than appear in the dividend; one, namely, for every cipher cut off from the divisor.*

Why not according to some excess of decimal places? — Integers only can have ciphers to be severed; there is therefore no question of excess of decimal places; nor does the severance of a cipher from an integer increase the parts, by

giving a denominator, but diminishes the whole by removing a place.

How may we know the value of the first place in the quotient of a decimal division? — If the decimals of the dividend taken on the first division do not exceed the decimals of the divisor, the first significant of the quotient is integral; if otherwise, its place is as far removed from units, as there are additional decimal places required for the first division.

What must be our first care in dividing? — Our first care in dividing is to see, that in the dividend there are decimals, significant or ciphers, sufficient for a first division; or, what is extremely convenient in long division, equal in number to the decimals of the divisor; for with them the integers of the quotient terminate.

What is the next? — Our attention must then be given to the true place of the first significant in the quotient, that, if removed more than one from units, ciphers may be prefixed without obliteration of the work that is to follow.

Can decimals terminated fractionally be used in division? — Decimals terminating in common fractions must either be reduced to approximates, or simplified, in order to become divisors; their dividends being increased by the same factor.

Can decimals be divided by common fractions? — Decimals are as easily divisible as integers by common fractions.

Can they be made the terms of common fractions? — Decimals are made terms of common fractions without creating the necessity of simplification.

How are repetends used in division? — As dividends, repetends afford the greatest advantage to be derived from their use, that of constantly presenting a true figure for annexation to remainders; in every other case they must be used as approximates, or reduced to common fractions.

Have you learned the rule?

RULE OF DECIMAL DIVISION.

Arrange and divide decimals like integers, extending the dividend, if necessary, by ciphers or repetends, to as many places as will suffice for a first division.

The integral places of a quotient continue, in significants or ciphers, as long as decimals taken of the dividend, or annexable thereto, do not exceed the decimal places of the divisor; its decimal places are one for every decimal of the dividend exceeding the decimal places of the divisor; and should the digits of the quotient fall short of this excess, ciphers must be prefixed to equalize them.

When ciphers are cut off from an integral divisor, the separator must be set, or the decimator advanced, an equal number of places toward the left; or the quotient must have as many decimal places additional to those corresponding with any decimal places in the dividend.

The place of the first significant of the quotient is integral, if the decimals taken of the dividend, for the first division, do not exceed the decimals of the divisor; if otherwise, its distance from units is the number of places exceeding.

Decimals fractionally terminated must be reduced to approximates, or may be simplified, in order to become divisors; their dividends being increased by the simplifying factor.

Repetend dividends must be continued, to the last division, by their proper recurring figures; in all other cases repetends are to be operated with as approximates, or reduced to common fractions.

Proof. Decimal division is proved by decimal multiplication.

Multiplication and Division.

APPLICATION.

1. What is the product of 1720×38·09?

```
    38·09              172,0 : 6551,4·8 = 38·09, proof.
     1720                      516
  --------                     ----
     7618                      1391
   26663                       1376
   3809                        ----
  --------                      1548
  65514·80, answer.             1548
  ========                      ====
```

It must be recollected, that to cut off a cipher is not to give a decimal place to a divisor; such a severance therefore from a divisor must be marked, in every case, by the separator, not by the decimator.

2. What is the product of ·625×·0046?

```
      ·625             ·0046 : ·002875 = ·625, proof.
     ·0046                      276
   -------                      ---
      3750                      115
     2500                        92
   -------                      ---
  ·0028750, answer.             230
   =======                      230
                                ===
```

The first division requiring one decimal place more in the dividend than is contained in the divisor, shows that the first significant

of the quotient is of the value of tenths. The annexation of a cipher to the last remainder gives a seventh decimal place to the dividend.

3. How often are 93·16 contained in 628?

93·16 : 628·00 = 6·74109, answer.
55896
———
69040
65212
———
38280
37264
———
10160
9316
———
84400
83844
———
556

6·74109
93·16
———
4044654
674109
2022327
6066981
———
627·9999444, proof.

The answer is a mixed approximate, carried no farther than the 5th place, as the 6th would have given a cipher. How nearly this comes to exactitude, may be judged of by the proof, deficient in less than a myriadth only.

4. What is the quotient from ·00728 ÷ ·26?

·26 : ·00728 = ·00028, answer.
52
———
208
208

·00028
·26
———
168
56
———
·00728, proof.

5. What is the product of $\cdot05 \times 116{\cdot}245\frac{1}{12}$.

$116{\cdot}245\frac{1}{12}$
·05
———
·05 : $5{\cdot}81225\frac{5}{12}$, answer.
———
$116{\cdot}245\frac{1}{12}$, proof.

6. What is the product of $\cdot56 \times 69{\cdot}034178\frac{6}{11}$?

$69{\cdot}034178\frac{6}{11}$
·7 × ·8 = ·56
———
$48{\cdot}3239249\frac{9}{11}$
·8
———
$38{\cdot}65913998\frac{6}{11}$, answer.

·7 : $38{\cdot}65913998\frac{6}{11}$
———
·8 : $55{\cdot}2273428\frac{4}{11}$
———
$69{\cdot}034178\frac{6}{11}$, proof.

Examples to be wrought, proved, and recited.

What is the product of

1. 63×1709·01? 104×560·94? 108×716·0715?
2. $\frac{1}{73}$×4559·10156? $\frac{1}{57}$×6307·3921? ·921×14500?
3. 323×·11677? $\frac{1}{156}$×·001564? $\frac{1}{69}$×908? 77×·89367$\frac{3}{5}$?

What is the quotient from

4. 6704·321÷132? 810·70493÷526? 7094·0159÷·8?
5. 84÷71·6? 3081·0561÷59·34? 7916÷·091?
6. ·06159÷·0097? ·781÷·0397? ·09549÷47?

What is the product of

7. ·872×·10946? ·6065×3998? ·072$\frac{3}{5}$×·7815?
8. 806·371$\frac{7}{8}$×·0493? 85·203718$\frac{3}{13}$×·006574?

Contractions.

The author has entertained a general dislike to contractions in arithmetic, from an impression, that, if remembered at all, they are remembered imperfectly, and that they almost necessarily entail incorrectness. The diffusive nature of decimals however is such as makes it imperative on us to adopt some mode, if any there be, of circumscribing their extension by limits seemingly natural, and practically true. The method and rule now to be described, though of much earlier date than Hutton, are taken from his Course of Mathematics, with a few alterations, made for the sake of greater security in operation, and ease of recollection; with the addition also of more than one *casus omissus*, and a demonstration, which Hutton has not. The facility, correctness, and beauty of the method in multiplication will so strongly commend it to the student, that, once known, it will never be neglected.

RULE OF CONTRACTED DECIMAL MULTIPLICATION.

To contract the decimal part of a product, reverse the order of the multiplier, setting the *units* of an integral or mixed factor under that place of the multiplicand which is the same with the lowest decimal to be retained in the product. Reject, in multiplying, all the figures of the multiplicand to the right of the multiplying figure, and set every first partial product in a column, one under the other exactly, *increased* by the tens that might arise from the nearest place omitted; by 1, namely, from the digit 5, to the product 14; from 15 to 24 by 2; and so for every additional ten, computed as from fives to fours. The decimal places of the product are known from *the units* of the multiplier.

When the multiplier has tens or more, and the multiplicand does not contain as many decimal places as are desired in the product, in addition to as many more as there are places in the multiplier above units, annex ciphers to equalize them.

When the multiplier is a pure decimal, the decimals of the product will be one place more than those of the multiplicand considered as extending to the farthest reversed decimal of the multiplier; and if the places obtained be fewer, ciphers must be prefixed.

APPLICATION.

1. What is the product of 27·14986×92·41035, retaining 6 places only of decimals in the product?

27·1498600	27·14986
53014·29	92·41035
2443·487400	13574930
54·299720	8144958
10·859944	2714986
·271499	10859944
· 8145	5429972
· 1357	24434874
2508·928065, answer.	2508·9280650510, proof.

2. What are the products of ·781 × ·0123, ·781 × ·123, and 1·781 × ·123, retaining only 4 places of decimals in each product?

·781	·781	·781	1·781	1·781
3210·	·0123	321·	321·	·123
781	2343	781	1781	5343
156	9372	156	356	21372
23		23	53	
	·0096063, proof.			·219063, proof.
·00960, ans.		·0960, ans.	·2190, ans.	

The latter examples are given for illustration of a principle only. It may be asked, of what value is the contracted mode, if it is to be proved by the operation at length? The answer is, that the contraction will be resorted to in operations to be proved from some special result. In contracted multiplication the decimal places are easily distinguished from the very first.

Demonstration. The reversed order of the multiplier cannot affect the entire product, if the separate products all maintain their due relative position. Accordingly, this is preserved, by carrying to the extreme right each first partial product that is noted; for whatever number of places, in the reversed series of the multiplier, a figure is advanced toward the left, at the same number distant from

itself to the right will its product commence. Therefore, unless changed by a failure of tens that would accrue on the right or an excess in computing them, the entire product will be the same as by the extended mode, with the omission of certain right hand places; and the tens from the places omitted are carried to the left, as accurately as may be, by the process described in the rule; derived, it is presumed, from an examination of results.

The first partial product by the units of the multiplier, obtained according to rule, is in the lowest decimal place of the contracted product, for it is carried to the extreme right, as is every succeeding first product; and this being in the lowest decimal place, there can be no more decimals in the product than are contained thus far in the multiplicand; for a multiplier in the place of units will give a higher place of decimals than any of the decimal multipliers to follow; not in value only, but also in position; for, by the rule, every separate product finds its true relative situation.

Further, because two decimal numbers, multiplied together, produce more decimal places than are contained in either factor, reckoning toward the left; therefore a decimal multiplier, consisting of a single figure, will produce one decimal place more to the left than the multiplicand contains; and under the rule the multiplicand is considered as extending only to the farthest right-hand place of a pure decimal multiplier reversed. Nor can succeeding decimal multipliers extend their products beyond the highest product of the first multiplier in any rank, neither in value nor position; for in value they are less; and whatever number of places advanced to the left by reversal, their first products are carried back the same number to the right; and this can be compensated by no inequality of digits, since every removal but a single place to the right is equivalent to a division by 10, and no digit is equal to 10.

RULE OF CONTRACTED DECIMAL DIVISION.

To contract the decimal part of a quotient, find the integral places as usual; and if there be no integral place, the first significant decimal, noted according to its true value from the left; then, if all the places of the divisor fall short of, or equal only, the decimal places still desired in the quotient, continue to divide as usual, till they exceed by one place the places that remain to be found; cut off that one place, take only the last remainder for a new dividend, and every succeeding remainder in the same manner, separating, before every new division, one other place from the divisor; and whenever constrained, from the smallness of the remainder, to separate two, annexing a cipher to the quotient; proceed thus till all the places of the divisor are exhausted, remembering to increase every first product in the manner directed by the rule of contracted multiplication.

Should all the places of the divisor *exceed* the number of

decimals desired in the quotient, after finding the integral places or first decimal, as mentioned before, separate at once the entire excess.

APPLICATION.

1. What is the quotient of 92·41035 : 2508·92806, retaining 6 places only of decimals?

```
92,4,1,0,3,5 : 2508·92806=27·14986,   92·41035 : 2508 92806=27·14985,
     18482070      [answer.                18482070   [proof in
                                                      [defect.
      66072106                              66072106
      64687245                              64687245

       1384861                              13848610
        924104                               9241035

        460757                               46075750
        369641                               36964140

         91116                                91116100
         83169                                83169315

          7947                                 79467850
          7393                                 73928280

           554                                  55395700
           554                                  46205175

                                                 9190525
```

Here the Doctor's own example fails at the 5th place of decimals; he has stopped at the 4th; yet the variation from exactness is so small, and the saving of labor so great, that the answer may well be taken to be true in practice.

2. What is the quotient from ·0076543÷·53191, retaining six places only of significant decimals?

```
·5,3,1,9,1 : ·0076543=·0143902,   ·53191 : ·0076543=·0143902, proof.
     53191           [ans.           53191

     233520                          233520
     212764                          212764

      20756                           207560
      15957                           159573

       4799                            479870
       4787                            478719

         12                             115100
         11                             106382

          1                               8718
```

Contracted decimal division is so obviously an abbreviation only of the extended mode as to need no formal demonstration. We have seen, as might have been expected, that it will sometimes fail from the mode of carrying the tens; but the failure is trifling, and the advantage, in many cases, great; the author therefore concludes the subject, with recommending the learner to return to the examples for practice in decimal multiplication and division, and to perform them anew, making the *proof*, in every case, by the contracted mode, to the extent of *not fewer than six decimal places;* for short of millionths results satisfactory to an inquisitive mind can seldom be obtained. By thus familiarizing himself, in the outset of his arithmetical career, with the contracted modes, he will not fear to make use of them in his progress. The contractions now explained reduce the process less than as they are taught and exemplified in the work consulted, because the present author cannot see what harm there is in a few additional figures, especially in the hours of a young student; because he is anxious for the correctness of the integral parts of these results; and because it is the desire only of determining, as it were, interminates, that induces him to recommend contractions at all.

ARITHMETIC.

PART III.

COMPOSITES.

Definition.

What is the subject of the next part of Arithmetic? — The third part of Arithmetic treats of composites.

What are fractions? — Fractions are parts either of a single or a collective unit; as of one, or of a dozen.

How may the parts of a collective unit be considered? — The units comprehended under one larger unit may be considered and computed as smaller values.

Have you an instance? — A cent is comprehended under a dollar in valuation; but it is no part of a real dollar, being itself an entire coin of a different metal.

Does this comparison extend to other things? — All things of inferior value, in the same kind, yet of independent existence, may be estimated and computed as parts, or fractions, of some greater unit; as pounds of a hundred weight.

In this case, how may they be represented? — Smaller values may be represented either as common fractions; or, provided their denominators are known, in the manner of decimals, by integers with some mark of separation.

What is meant by different denominations? — By numbers of different denominations are meant the numbers representative of different and smaller values, together composing some one larger value; receiving consequently different denominators, and named therefrom.

Can you now tell me what are composites? — *Composites are units distributable into parts, or smaller values, having invariable denominators and peculiar names; and operated with as whole numbers. The lowest of such parts is also called a composite, from relation to the higher.*

What is the use of composites? — Composite numbers represent the prevailing divisions of money, weights, and measures; exhibited commonly in tables.

Can you state more particularly the nature of these tables? —Tables of money, weights, and measures, present, at a single view, the various modes of estimating value, quantity, and magnitude, in use among nations.

What constitutes the variety of mode? — Some modes are *material*, as coin, metallic weights, wooden and metallic measures; others, computative only, as a mill, a mile.

How are these varieties adjusted? — Some one value, in the same kind, being made the standard unit, inferior, and even superior, values are estimated proportionably to this; as the eagle to the dollar.

Money.

What is money? — Money is some intrinsically valuable medium of exchange and account, universally applicable wherever law or usage has established it.

What is the usual medium? — The universal medium adopted by civilized nations consists of pieces of metal, denominated coins, prepared and stamped by public authority.

Then what is coin? — Coin is the legal, metallic, stamped money of civilized nations.

Have they no other description of money? — Computative has always a reference to material money, and may usually be discharged therein.

It is to be lamented, that governments do not use the *golden* opportunities they have of recording the great events of their national history, by converting every coin into a medal; a source of infinite instruction might thus be created.

Coinage and substitutions.

What is alloy? — Alloy is a portion of inferior metal mixed with silver and gold, to increase their tenacity, and thereby lessen the effect of friction on coins.

What is base coin? — Base coin is silver and gold money alloyed more than is necessary for fair coinage.

When is such money made? — Base money is never coined but by weak rulers, who think to enrich themselves, or to increase the resources of the state, by making coin to pass for more than it is worth.

What prevents it? — The artifice is soon found out; and either the money is depreciated in calculation, or commodities are raised in price.

What is counterfeit money? — Counterfeit money is that coined by sharpers, who use their skill to imitate the dies of

the government, and the metals esteemed precious, that they may live a life of profligacy at the expense of their fellow citizens, especially of the working class, who are the chief sufferers by such arts.

Are metals the only medium of exchange current among men? — In a few half-civilized, and some savage, communities, other means of a convenient and general barter have grown into use; as the cowries of the Hindoos, and the wampum of North American Indians.

What are these? — Cowries are small shells, used as divisions of the meanest metallic coin; wampum is a bead manufactured of shell.

How may such means of barter be characterized? — These are substitutes for money, having in themselves no intrinsic value whatever; whereas metals must ever have a highly important value, even could money be banished from the earth.

Are there no substitutes for money to be found among improved nations? — The great increase of commerce, with the comparative rarity of gold, has led to the substitution of bank notes.

What are these? — Bank notes are partly printed, partly written, engagements to pay at sight, during the accustomed office hours, the sums expressed on them; their current value depends entirely on the credit given to the issuers.

Are not these also counterfeited? — The temptation to forgery offered by bank notes has been a source of great calamity to individuals and families.

How should the crime be estimated? — Forgery is among the most atrocious crimes that can be committed against the well-being of civilized society; for all the more important transactions of human life, and a very large portion of every day's concerns, are now, in some way or other, connected with written engagements.

Is this an evil attributable to banking institutions? — Few advantages can be acquired in human life without some attending evils.

What then are the advantages derived from banks? — Banks established by honorable and responsible men afford secure places of deposit, by which the crimes of murder and robbery must, in their own nature, be greatly diminished; and they offer facilities to commerce and travel, without which the energies of men would stagnate and die.

Are all such institutions of this character? — The profits arising from banks have often induced needy and crafty adven-

turers to engage in them; and their mischievous operations have, at times, spread desolation through whole communities.

Names and Materials of Money.

What are the different names given to money? — Metallic money has the different names of coin, cash, and specie, as it is often called.

What is specie? — *Specie* is a Latin word, in an oblique case, signifying, *in kind;* the nominative case *species,* for money, seems to have gone out of use.

What are the metals called precious? — The precious minted metals are gold and silver.

What is bullion? — Bullion is any precious metal uncoined.

Are you acquainted with the relative values of these metals? — The value of gold has been, during some years, about 14½ times that of silver, both rated at the standard purity.

What is that standard? — American and British gold coins have, each, one part alloy in twelve; United States silver coin has the same; British silver coin one part alloy in thirteen parts and one third.

Does this affect mercantile transactions? — It occasions a difference in the estimated value of equal weights of American and British silver coin; and at times a rate of exchange apparently disadvantageous to the United States, when it is actually in their favor.

MONEY OF THE UNITED STATES.

What is federal money? — Federal money is the coined and computative money of these United States; called federal, because established by the federative government.

How is computative money usually named? — Money of account, that is, of account only, is the name commonly given to money denominated but not coined.

What are the coins of the United States? — The coined money of the United States are the eagle and half eagle, of gold; the dollar, half dollar, quarter dollar, dime, and half dime, of silver; the cent, and half cent, of copper.

What the money of account? — Mills are the only imaginary money of the United States.

According to what arithmetical scheme is federal money arranged? — The divisions of federal money are decimal, the coinage of halves and quarters of a dollar, making a difference only of expression, not of notation and account.

Whence have the lower denominations their origin? — *Dime* is a French word, signifying a tenth part; with us,

namely, of a dollar; cent is a diminutive, from the Latin word, *centum*, a hundred; mill, from the Latin word, *mille*, a thousand.

Are accounts kept in all these divisions? — Accounts of federal money are kept in dollars, cents, and mills; often in common fractions of cents.

Has this mode of computation any peculiar advantage? — The decimal division of money was adopted, on account of the great facility of decimal computation.

Is the advantage realized? — The advantage could not fail of being realized in regard to federal money itself; but the estimation of the dollar in parts of the ideal pound sterling, and especially its varying estimation in different States, render American computations not less perplexing than European.

Can you state some of these varieties? — In New England, Virginia, Kentucky, and Tennessee, the dollar is valued at six shillings; in New York and North Carolina at eight shillings; in New Jersey, Pennsylvania, Delaware, Maryland, and Ohio, at seven shillings and sixpence; in South Carolina and Georgia, at four shillings and eight pence.

Are there not other causes of variation from the decimal scheme? — The circulation of the silver coins of South America, and of Mexico, occasion also a diversity of computation.

What are the chief of these coins? — The smaller Spanish and Portuguese coins are, the pistareen, worth about 17 cents; the rial, or bit, worth 12½ cents, of which eight go to the dollar, or Spanish piastre; the half bit, worth 6¼ cents.

Can you now recite the table of federal money, coined and computative?

FEDERAL MONEY.

10 mills	=	1 cent.
5 cents	=	1 half dime.
10 cents	=	1 dime.
10 dimes, or 100 cents	=	1 dollar. (Marked) $.
5 dollars	=	1 half eagle.
10 dollars	=	1 eagle.

Accounts are kept in dollars, cents, mills, or common fractions of a cent.

Standard of gold and silver, 22 parts fine, 2 alloy.

Weight of the eagle, 10 *dw.*, 6·26 *gr.* of fine, 11 *dw.*, 6 *gr.* of standard, gold.

Mint price of standard gold, $209·77, the pound Troy.

Weight of U. S. dollar, 15 *dw.*, 15·64 *gr.* of fine, 17 *dw.*, 10 *gr. of standard,* silver.

Value of U. S. dollar in British money; estimated, 4*s*. 6*d*.; real, 4*s*. 3¾*d*.

Value of the pound sterling, at 4*s*. 3¾*d*. to the dollar, $4·6376. (The author suspects this to be the true value.)

Weight of copper in 100 cents, 2¼℔. avoirdupois.

Value of a cent in British money, ·5263*d*.

The sign of equality may be read in all the tables by the verbs make *or* equal.

STERLING MONEY.

Has British money any other appellation? — British money is also called sterling.

Whence this word? — Sterling is from Esterlings, certain German artificers who first coined silver pennies, thence called esterlings, for the British monarch.

Has sterling always the same signification? — Sterling has always a reference to British money, but not always according to its estimation in Britain; for the different valuation of the dollar seems to be derived from a comparison of the ancient colonial currency, varying in different colonies, with the standard unit of Britain.

What is the coined money of Britain? — The coins of Great Britain are, the farthing, or fourth thing, the halfpenny, and the penny, of copper; the sixpence, the shilling, the half crown, and the crown, of silver; the sovereign, the half sovereign, the guinea, and half guinea, of gold.

What is the imaginary money of Britain? — The only money of account in Britain has been the pound sterling; this however was coined in the reign of George IV. under the denomination of sovereign.

Whence has the former denomination its origin? — The pound is so called from having anciently been a troy pound of silver in material.

Can you now recite the table?

BRITISH MONEY.

4 farthings	=	1 penny.	(Marked) *d*.
12 pence	=	1 shilling.	*s*.
5 shillings	=	1 crown.	
20 shillings	=	1 pound or sovereign.	£.
21 shillings	=	1 guinea.	

Accounts are kept in pounds, shillings, pence, and farthings.

Standard of gold coin, 22 carats of fine gold, 2 carats of the purest copper.

Standard of silver coin, 11 *oz*. 2 *dw*. of fine silver, 18 *dw*. of *alloy*.

Weight of the guinea, 5 *dw.* $9\frac{39}{89}$ *gr.* of standard gold; mint value of a pound of standard gold, $44\frac{1}{2}$ guineas.

Weight of the sovereign, 5 *dw.* 3·274 *gr.* of standard, 4 *dw.* $17{\cdot}0011\dot{6}$ *gr.* of fine, gold.

Mint price of standard gold, per oz. £3 17*s.* $10\frac{1}{2}$*d.*; of silver, 5*s.* 2*d.* per oz.

1lb. troy of standard silver is coined into sixty-six shillings.

Value of a pound sterling in federal money, reputed, $$4{\cdot}44\dot{4}$; by act of Congress, 1832, $4·80.

Value of a penny sterling in federal money, 19 mills.

Decimal values in the Pound Sterling.

1 farthing = £$\cdot 001\frac{1}{24}$
1 penny = $\cdot 004\frac{1}{6}$
3 pence = ·0125
6 pence = ·025
1 shilling = ·05
2 shillings = ·1

N. B. In reducing sterling to decimal values, it will often be convenient and not unjust, to note an inconvenient terminating common fraction as a decimal of the next highest or lowest value; a thing which, after a little use, may be done in an instant, but should never be done with manageable fractions.

FRENCH MONEY.

What are the coins of France?—The modern coins of France are, of gold, the twenty and forty franc piece; of silver, the five, two, and one, franc piece; the half franc or fifty centimes, and the quarter franc of 25 centimes; of copper, the decime and the sou.

Which is the money unit?—The franc is the money unit of France, every other coin being a multiple or measure of it.

Can you recite the table?

FRENCH MONEY.

5 centimes = 1 sou.
10 centimes = 1 decime.
10 decimes = 1 franc.
20 francs = 1 louis?
100 francs = 101 livres tournois and 5 sous.

Accounts are kept in francs and centimes.

A franc is equal to $$\cdot 18\frac{3}{4}$; five francs to $$\cdot 93\frac{3}{4}$.

Tournois is used of ancient French money, as sterling is used of British money.

Weight.

What is weight? — Weight is the tendency of all earthly bodies to approach the earth's centre.

Do they all thus tend in an equal degree? — In spaces void of air, as shown by experiments with the air-pump, all bodies gravitate alike, or descend with equal rapidity, the feather and the coin.

What are the ordinary circumstances affecting gravitation? — The resistance of the air causes equal bulks to descend with a velocity proportioned to their density.

What is density? — Density is the comparative quantity of matter contained in equal bulks of different things; as in a coin and a piece of wood of precisely the same dimensions.

Then what is weight applicable to? — Weight is applicable to density alone.

What are weights? — Weights are masses of metal, used to proportion commodities to price, and one commodity to another.

Are there no other means of determining this proportion? — Since the different gravitation of bodies is the same with the different velocity of their descent to the earth under like circumstances, an instrument has been used for ages which determines weight by velocity.

Of what nature and name? — The instrument is called a steelyard; it is a lever, or small iron beam, unequally divided into two arms, supported at their line of union, and so adjusted, that an inconsiderable weight on the longer arm shall counterpoise a far greater weight suspended on the short arm, at a small distance from the centre of gravity and indicate its amount.

What is the principle of this adjustment? — The short arm, by its massiveness and hooks is made a counterpoise to the longer arm, independently of the weight, called a P (the initial of pound or power); when therefore the weight of a commodity and of the P are exactly equal, this will be shown by the P's forming an equipoise at the same distance from the centre of gravity on one side, with that of the commodity on the other; and at a distance as much nearer or farther off, as the commodity is of less or greater weight than the P.

How is velocity concerned in his? — Velocity is as the tendency to descend; and the removal of the P to different parts of the beam varies that tendency; for it is a power acting on a longer or shorter lever; but, the longer the lever, the greater is the velocity, for it describes a larger arc in the same time *that a shorter* lever would move through a smaller arc.

What are scales? — Scales are a lever of equal arms, having each a dish, shallow or deep as occasion requires, attached to it; the arms of the lever being equal, their tendency downward is equal, and the balance is still preserved, by making the opposing dishes and fastenings of exactly the same weight.

When are the steelyard and scales known to be in equipoise? — When the tongue, as it is called, of the instrument is perpendicular to the point of suspension, the balance is true, though the weights or the graduation may be false.

Which of the two is the most accurate balance? — The steelyard is fit only for weighing the grosser commodities, in which a trifling variation from exactness is of no importance.

What are the weights used among us? — The divisions of weight employed in the United States and Britain are three, denominated troy, apothecaries', and avoirdupois, weight.

What was the origin of these weights? — The original standard of weight in England consisted of thirty-two dried grains of wheat taken from the middle of the ear: these were called a penny weight (perhaps from the weight of a silver penny); but the penny weight having been once formed, it was thought more convenient to divide it into twenty-four, than into thirty-two, parts, though still retaining their original name of grains.

Has any thing of late been done affecting the ancient standards of weight and measure? — In the year 1825, an act was passed by the British Parliament, to take effect from the following 1st of January, amending the ancient standards, and establishing, for their country, a uniform system of weights and measures.

The amended standards are those of the following tables, which, in their outline, are the same as hitherto, yet really both simplified and made more complete.

PLATE.

Whence has troy weight its name? — Troy weight is supposed to be so called, from having been introduced from Troyes, a town of France.

What are the articles estimated by it? — By the troy table are estimated all precious stones, except diamonds; and by the same table, bullion, gold and silver plate; bread also, and liquors when weighed.

What is the unit? — The standard unit of all weights is, by the British act, ordained to be the Troy pound of five thousand seven hundred sixty grains, a proportion derived from the weight of a cubic inch of distilled water, at 62° of Fahrenheit's thermometer, and 30 inches of the barometer; the weight of which is also declared to be, 252·458 troy grains.

Has any nation a standard purity for plate? — The British standard of silver plate is the same with that of U. S. coin; namely, one part alloy in twelve.

What of gold? — The British standard of gold plate is also the same with that of U. S. gold coin, watch-cases excepted: these are allowed to be of eighteen cáracts fine.

How is the observance of this standard enforced? — There is, in London, an assay office of the government, to which all plate must be sent, and where it will not be stamped with the office marks (marks that are always distinct and elegant), unless it prove on trial to be of the standard purity.

Does the regulation extend to every thing made of gold and silver? — Trinkets that will not bear a mark are subjected to no test; the quality therefore of their material is altogether uncertain.

Whence is the mark of the pound weight derived? — We derive the mark ℔ from the Latin word *libra*, a pound weight.

Can you recite the table?

TROY WEIGHT.

24*gr.* grains	= 1 pennyweight.	(Marked) *dw.*
20 pennyweights	= 1 ounce.	*oz.*
12 ounces	= 1 pound (5760 grains.)	℔.

The pennyweight for pearls is divided into 30 grains; hence 4 troy grains equal 5 pearl grains.

DIAMONDS.

What is a cáract? — A cáract is one twenty-fourth part of any weight, but ordinarily used of small weights.

To what is the term itself usually applied? — The term is almost exclusively applied among us in estimating the proportion of pure gold to the alloy in any mass of the precious metals, and in determining the weight and value of diamonds.

Are the divisions of it the same in both cases? — The minter's cáract is divided into twenty-four parts, the diamond dealer's into sixteen.

How are diamonds valued? — The value set on cut diamonds increases as the squares of their weights multiplied into the price of a single cáract.

Can you exemplify this? — If a diamond of a single cáract weight be estimated at £8 sterling, one of two cáracts will be valued at four times that price, or £32; one of three cáracts, at £72; so to the weight of twenty cáracts.

Why do you stop there? — Beyond twenty cáracts, so few purchasers are found, that prices become arbitrary.

What is the table?

DIAMOND WEIGHT.

16 parts	= 1 diamond grain,	or ·8 *gr.* troy.
4 diamond grains	= 1 cáract,	or 3·2 grains troy.

What are the moneyer's divisions? — The gold ounce, so called, to distinguish it from the Troy ounce, is divided at the mint into twenty-four cáracts of the standard purity; twenty-two of these are of fine gold.

What is the table?

MINT MASTER'S TABLE.

4 grains	= 1 cáract.
24 cáracts	= 1 gold ounce.

Cáract is probably derived from *cárecta*, (Rees), a low Latin word, used for any weight; considering its accentuation, to spell and pronounce it cárat, makes it a very ridiculous word.

PHARMACEUTICS.

In what does apothecaries' differ from troy weight? — Apothecaries' weight differs from troy in an adaptation only of some of its divisions to medical purposes; the pound, ounce, and grain, being the same in both.

What are the chief uses of this table? — Medicines are compounded and sold by apothecaries' weight; but drugs in quantities are bought and sold by avoirdupois.

What is the meaning of the word drachm? — Drachm is from *drachma*, the Latin name of the eighth part of an ounce.

Can you recite the table?

APOTHECARIES' WEIGHT.

20 grains (troy)	= 1 scruple.	(Marked) ℈.
3 scruples	= 1 drachm.	ʒ.
8 drachms	= 1 ounce.	℥.
12 ounces	= 1 pound.	℔.

GROCERIES.

What is the signification of the word avoirdupois? — *Avoir du pois* is a French phrase, signifying, *to have weight;* by which is meant, considerable weight.

To what uses are the table applied? — Most kinds of bulky and heavy commodities, groceries, anthracite, flour, tobacco, all metals except the precious, are estimated by avoirdupois weight.

Is there any agreement between troy weight and avoirdupois? — The divisions both of troy and avoirdupois weights are derived from a dried grain of wheat, taken from the middle of the ear; their grains of account therefore are the same.

Where is the diversity? — The troy ounce is the heaviest, weighing 480 grains; the ounce avoirdupois only $437\frac{1}{2}$ grains; yet the pound avoirdupois is much the heaviest, from containing a greater number of ounces.

Are the denominations and divisions of avoirdupois uniform? — In the United States the hundred weight is estimated, we believe generally, at an exact hundred pounds; the quintal, or great hundred of 112℔, being used chiefly in transactions connected with British commerce; yet retained also in the weighing of fish.

Do not varieties of division occur also in different classes of commodities? — In some lines of greater merchandise, packages are ordinarily of the same bulk and weight, and thence receive some peculiar denomination; all however are estimated from the pound avoirdupois.

Can you recite the divisions?

AVOIRDUPOIS WEIGHT.

27·34375 grains troy	= 1 drachm avoirdupois. (Marked)	*dr.*
16 drachms	= 1 ounce.	*oz.*
16 ounces (or 7000*gr.*)	= 1 pound.	℔.
25 pounds	= 1 quarter American.	*qr.*
28 pounds	= 1 quarter British.	
100 pounds	= 1 hundred weight, Amer.	*A cw.*
112 pounds	= 1 quintal, or hund. w't Brit.	Q.
4 quarters	= 1 hundred weight.	
20 hundred weight	= 1 ton.	T.

Measure.

What is measure? — Measure is a general expression of estimated quantity; of the means also used in making such an estimate.

What great division is measure susceptible of? — Measure is divided into measures of weight, or gravity; of extension; and of motion, or, considered in a peculiar manner, of time.

Space.

What other name have measures of extension? — Measures of extension are also called measures of space.

Correctly, or otherwise? — We apprehend not correctly; *because* space is infinite, therefore immaterial, indivisible, and *immeasurable.*

How then would you define space? — Space is that infinite and immaterial subtance in which all finite and material things exist.

Why should it be supposed infinite? — Space is infinite; for it is everywhere; it bounds every thing, and is bounded by nothing.

Why immaterial? — Space is immaterial, for it does not gravitate, being everywhere equable; and being infinite, if it were material also, all things would be full; there could be no fluid, no motion in the universe; all would be hard and unyielding as the rock.

What is measured then? — We measure, not space, but the extension of material things, or the distance of one from another, by the interposition of a third material thing.

Do we never hear of the distance of the stars? — The distance of the heavenly bodies is computed, by imagining a line, divided into portions, and extended between the bodies whose distance from each other is computed; but we cannot think of a line without something material to form it, or to bear the impress of it.

Is not space itself extended? — It is impossible for us to say of any thing immaterial, whether it be extended or not; but this we know, that, with the disappearance of material objects, our idea of extent is very much lessened; as on the ocean, when neither land nor vessels are in sight.

Since however gravity is measured, why cannot we measure space? — The affirmation is, that space never is measured, but only the extension of material things, or their possible extension; beside which, gravity is a power finite in its action; space is a substance infinite in its nature.

Is it any thing? — Space must be something; for if nothing intervened, we should be in contiguity with the sun, moon, and stars; which everybody knows to be false.

May not our atmosphere extend so far? — Beside the absurdity of imagining that the atmosphere of a globe about eight thousand miles in diameter should extend through the universe, the probable extreme height of it has been determined.

After all, do we know any thing of these matters? — Because we do not know every thing, it is the constant refuge of silliness and skepticism, when foiled in argument, to say, we know nothing.

Whence do you infer this? — The weight of a column of air descending from its extreme height is known by its pressure on the barometer; it is known, also, by observations on

the same instrument, that the air grows thinner as it ascends; it must at length therefore be attenuated to nothing.

What answer does this afford to the materialist? — To affirm that all things are material, is to contradict the evidence of our senses as well as of our reason; for we have the strongest possible evidence from both, that we live and move in a region of immateriality.

Bishop Gleig, in one of his metaphysical dissertations (Ency. Brit. 3d ed.) contends, and with a slight alteration of language it would seem to be no wonderful discovery, that space is an idea only of the possible existence of body. Now this idea is either from without, a supposition implying that space is a reality; or it must be supposed innate, a Berkleian whimsey, dismissed at once, when we recognise the absurdity of attempting to prove first principles. He contends further, that although things in contact have nothing between them, it does not thence follow, that things *not* in contact have something between them. The bare statement of such a proposition is ridiculous; since the existence of something or nothing between divides the whole ground of distinction; and if things can be remote from each other, yet have nothing between, the absurdity follows, that to be in contact, and not to be in contact, denote the self-same condition.

In its most usual application, what is measure? — Measure, as commonly understood, is the estimate of one extension by comparison with some other, considered as unity; as of any larger extent by the foot, or the ell.

How was this comparison first ordered? — The measures of England may have had an original similar to its weights; these from grains of wheat; those from grains of barley; three barleycorns, placed lengthwise and in contact, forming the first denomination of English linear measure.

Do not the denominations of hand and foot indicate a different origin? — As the fingers would first suggest the idea of number, some other part of the body, the foot, for instance, may have suggested a standard of measure; while, for its subdivisions, sought in some natural standard also, grains of barley may have been chosen.

In how many ways is measure applicable? — Measure is applicable to one dimension, of length, namely; called long, or linear, measure; to two dimensions, namely, length and breadth, or superficial measure; to three dimensions, length, breadth, and height, or depth, called cubic measure.

MEASURES OF CAPACITY.

What is meant by capacity in respect of measure? — Capacity, or capaciousness, is the power of containing.

What are the measures denominated from it? — Measures

of capacity are hollow vessels of fixed dimensions, made use of to estimate certain commodities, both dry and liquid, and not usually weighed.

STRICKEN MEASURES.

What are the measures of the former kind? — The measures of dry commodities, are grain measure, and coal, or heaped, measure.

What articles are usually arranged under the first of these tables? — Seed of all kinds, salt, when not sold by weight, and whatever is stricken off smooth from the measure, are sold by grain measure.

What is the mode of striking? — The striking must be by a light piece of round wood, and the measurer cannot be required to shake the measure.

Are the farinaceous grains included in this measure? — Seeds comprehend grain; wheat, however, of the same bulk, but of greater weight, being much the most valuable, is now frequently sold by weight.

What is the standard? — The standard of all measures of capacity is the gallon.

What are its cubic contents? — The contents of the grain gallon have hitherto been 268$\frac{4}{5}$ cubic inches; the British standard, for liquids, and for solids smoothed off, is now of 277$\frac{1}{4}$ cubic inches.

Can you recite the table?

GRAIN MEASURE.

34$\frac{21}{32}$ cubic inches	= 1 pint British.	(Marked) *pt.*
2 pints	= 1 quart.	*qt.*
4 quarts	= 1 gallon (277·274*ci.*)	*ga.*
2 gallons	= 1 peck.	*pc.*
4 pecks	= 1 bushel (10$\frac{1}{4}$*cf.* nearly).	*bu.*
8 bushels	= 1 quarter.	*G qr.*
5 quarters	= 1 load.	

The British standard grain bushel contains eighty pounds avoirdupois of pure water, exactly.

How does grain measure differ from heaped measure? — Heaped differs from grain measure in all its dimensions, and some of its denominations.

What are its uses? — By heaped measure are estimated bituminous coal, charcoal, lime, potatoes, fruit, and whatever else is customarily heaped above the measure.

How is the heaping regulated? — The British statute re-

quires, that such articles shall be heaped up conically, to a height above the rim of not less than three fourths of the depth of the measure; and, by another law, the coal bushel must be edged round with iron.

What are the divisions?

HEAPED MEASURE.

352*ci.* (nearly)	=	1 gallon.
2 gallons	=	1 peck.
4 pecks	=	1 bushel.
3 bushels	=	1 sack.
36 bushels	=	1 chaldron.

Depth of coal bushel within, 9¾ inches; diameter outside, 19½ inches; breadth of the edge three eighths of an inch.

Depth of fruit gallon within, 4⅞ inches; diameter outside, 9¾ inches; breadth of the edge, three sixteenths of an inch.

Weight of a chaldron of coal anciently, 1 ton English; now said to vary, from 2800℔, to 3156℔, avoirdupois.

Measure of 2000℔ of anthracite, 25 bushels, or thereabout.

LIQUIDS.

What is the remaining measure of capacity? — The measure for liquids has hitherto differed, in all its dimensions, and in many of its denominations, from that for solids; but the British statute of uniformity in weights and measures has made the gallon, of one and the same capacity, the standard measure both of liquids and solids, those only excepted which fall under heaped measure.

How far are the denominations applicable to both? — From pints to gallons, the divisions and denominations of grain and liquid measure are the same; after gallons they necessarily differ, on account of the very different size and shape of the containing vessels.

Have those denominations any legal recognition? — Beyond gallons, the denominations in liquid measure have only a customary and convenient use; the contents of every cask being always determined by the gauge and the gallon.

What distinctions exist between the measures of wine and beer? — The distinctions that still exist in wine and beer measure are in the size and denomination only of casks used in the respective trades; the former difference of standard being now legally abolished.

Can you recite the table?

LIQUID MEASURE.

$34\frac{2}{3}\frac{1}{2}$ cubic inches	= 1 pint British.		(Marked) *pt.*
4 (*gi.*) gills	= 1 pint.		
2 pints	= 1 quart.		*qt.*
4 quarts	= 1 gallon.		*ga.*
$31\frac{1}{2}$ gallons	= 1 barrel.		*brl.*
48 gallons	= 1 hogshead of ale.	}	
54 gallons	= 1 hogshead of beer.	}	*hhd.*
63 gallons	= 1 hogshead of wine.	}	
2 hogsheads	= 1 pipe.		*p.*
2 pipes	= 1 tun.		*L t.*

The British standard gallon contains exactly ten pounds avoirdupois of pure water; the pint consequently, $1\frac{1}{4}$ ℔.

LENGTH.

To what use is linear measure applied? — Linear measure is used for determining a single dimension, or length.

What is the difference between length and breadth? — Length is the longest, breadth the shortest, dimension of the same surface.

How then does linear differ from superficial measure? — By linear measure we ascertain the length of lines only; by superficial measure we ascertain any extent of surface circumscribed by line, or lines.

What was the original of British linear measure? — It is difficult to say precisely what was the original of English linear measure; but, from the denomination and general prevalence of the term foot, this has probably been the standard in many countries.

When is the first distinct notice on this subject? — In Henry the First's time, the yard was established; taken, it is supposed, from the length of the king's arm.

What is now the standard? — The present British standard is still the ancient yard, which is subdivided into feet and inches; and the length of which is nine hundred nineteen thousand seven hundred ninety-two millionths of a pendulum vibrating seconds of mean time in the latitude of London, at the level of the sea, in a vacuum.

Are there subdivisions still lower than inches? — The inch has been divided into eighths and twelfths, but these are going out of use, the decimal division being now exclusively adopted by officers of the revenue in Britain, and by men of science.

What is understood by a great circle of the earth? — A great circle of the earth is a line investing its largest dimen-

sions: these are at the equator, in the direction of east and west, and through the poles all round, in any line north and south.

The author has before him a rule made by an eminent mathematical instrument maker of London, having the decimal division of the inch; these divisions he shall call by the former name of lines, and the least subdivision, points, answering to tenths and hundredths, as low perhaps, as it is possible to go in practice, though in computation we are unconfined. The mark of the line he imitated from those of duodecimals, but leaning to the right to distinguish linear from square measure.

Can you recite the table?

LINEAR MEASURE.

10 points	= 1 line.	(Marked) `	
10 lines	= 1 inch.	*i.*	(12 lines = 1 inch, [in duodecimals.)
4 inches	= 1 hand.		
12 inches	= 1 foot.	*ft.*	
3 feet	= 1 yard.	*yd.*	
2 yards	= 1 fathom.		
5½ yards	= 1 rod.	*r.*	
40 rods	= 1 furlong.	*fu.*	
8 furlongs	= 1 mile.	*ml.*	
3 miles	= 1 league.	*lg.*	
69·07 miles	= 1 degree of a great circle of the earth,	*E* °.	

The entire length of the pendulum referred to is declared by statute to be, thirty-nine inches, thirteen hundred ninety-three myriadths.

What peculiar measures fall under the description of length? — Peculiar linear measures are the span, the cubit, and the pace.

What is the first-mentioned? — A *span* is the supposed stretch of a man's hand, from the tip of his thumb to that of his middle finger; estimated at nine inches.

The next? — A *cubit* is the supposed length of a man's arm and hand, from the elbow to the tip of the middle finger; estimated differently at different times; from eighteen inches, the English cubit, to the Scripture cubit of nearly twenty-two inches.

The third? — A *pace* is the supposed stretch of a man's legs, from the hindmost heel to the forward toe; estimated at five linear feet.

The computation would certainly seem to be of men of olden time.

What are the surveyors' means of admeasurement? — Surveyors use a metallic chain of fixed length, for determining their lines.

Are you acquainted with its divisions?

SURVEYORS' CHAINS.

7·92 inches	=	1 link.
25 links	=	1 rod.
100 links	=	1 chain.
10 square chains	=	1 acre.
80 chains	=	1 linear mile.

The measures now adopted in Great Britain may be considered, from their reference to a philosophical standard, as possessing a permanent character, for that country. Yet, if a uniform system of measures were to be devised for the United States, it would seem desirable to make the philosophical standard itself the standard of measure. In all improvements of this kind, two objects are to be aimed at, permanency, and the least possible innovation. It would be extremely difficult, if not impossible, to introduce here all the divisions, and any of the names of the French system; and the length of the seconds-pendulum approaches so nearly to the French *metre*, their difference being much less than an inch, that if the former were adopted as the standard, the two measures might safely be identified in the loose calculations of general reading. If the author may be indulged in such a speculation, he would hazard the following arrangement and terminology of a new system; to the word *dime* in measure, he thinks there can be much less objection than to the word *pound* in weight; dime being used for money in books only.

	10 points	= 1 line.	
	10 lines	= 1 dime.	
	10 dimes	= 1 foot.	(Marked) ⌣
	3 feet	= 1 rule, or seconds-pendulum.	—
In 38° 53′, the lat. of Washington, 5·06 rules, probably,	5·0589 rules	= 1 rod.	r.
	40 rods, &c. as hitherto.		

By taking the above decimal of the proposed *rule*, there would be an increase in computation, of less than $\frac{3}{5}$ of an inch in a mile, in lat. 51° 51′; an inaccuracy, one would imagine, that could not be felt in practice, and which might be corrected in the degree. The new foot would a little exceed thirteen of our present inches. Insufficient as the objections are elsewhere to a decimal division of the foot, they could hardly arise among a people accustomed to a decimal division of money. Beyond the foot, no real advantage would be obtained from a decimal division; *within it* are comprehended all the important calculations now performed with duodecimals, the abandonment of which, by the substitution of decimals, would give a vast facility to business. The only superiority of the duodecimal above the decimal division is, its enabling us to take a third.

SUPERFICIES.

What is a square? — A geometrical square is a figure of two equal dimensions, the sides of which are perpendicular or parallel, each one to every other.

What is the square of a number? — An arithmetical square is the second power of any number; or first product obtained from a number multiplied into itself.

Why is the same name used of both form and number? — If a square figure be divided into any number of equal and smaller squares, of the dimensions denominated, that number will be found to be the same with the second power of the sum of the squares on any one side, reckoning a unit to each; hence the sameness of appellation, for the number truly represents the thing.

Have you an instance of it? — The multiplication table in squares is an instance; for the 2d power of any tabular number will there be found equal to the sum of all the squares comprehended within four equal adjacent lines, two of which extend to that number; and the sum of any one of the sides to be equal to the first power.

Cannot the same thing be demonstrated from reasoning? — Let one factor represent a line of smaller squares, in number equal to its units, and of dimensions answering to the denomination specified; this line, taken once for every unit, since both factors are the same, may form a square figure; for there will be as many lines of squares as there are squares in a line; the sides therefore are equal.

Suppose the factor to be a fraction? — If the factor be a fraction, the plane will then be a portion of that which is supposed to be represented by unity.

Can you exemplify this? — The $\frac{1}{2}$ of a foot square, multiplied into $\frac{1}{2}$, produces $\frac{1}{4}$ of a foot square, or 36 square inches, the 4th part of 144.

Would not fractional multiplication thus seem to be increase? — Fractional multiplication is always diminution; but in the raising of fractional powers, if such a phrase can justly be used, we must assume the fraction itself to be in the power sought; and, by multiplication, take the required portion of it.

How does this appear? — It appears manifestly from taking the half of half a foot; for if one of the halves be not assumed as of a foot square, the product must be $\frac{1}{4}$ of a linear foot; fractional multiplication being necessarily diminution.

Does involution afford any further illustration of this? — ·e unit is assumed, or inferred, to be of any power we

please, and can be raised by inference or assumption only; since one taken once receives no addition.

What is the conclusion? — If the unit must be assumed as of a power, because we cannot increase the notation of the same digit; still more must a fraction, since, by involution, we can only diminish its notation.

How is the extent of a surface ascertained? — Superficial extent is ascertained by square measure.

Are all surfaces square? — Surfaces are of every form.

How then do you obtain any superficial extent? — *To find the square dimensions of any regular four-sided plane, multiply the length into the breadth.*

This produces what? — The product represents a square of dimensions equal to the superficies under consideration.

How is this demonstrated? — The product truly gives the actual dimensions; for either factor may represent a line of smaller equal squares of the highest denomination specified, and in proportion for any parts; this line may be taken once for every unit in the other factor; there will consequently be as many smaller squares in the figure as are equal to the product of one factor by the other. [See multiplication table in □s.]

How demonstrated that it represents a square? — Every number may be considered a square, perfect or approximate; since a factor may be found, which, multiplied into itself, will produce that square, or approximate very nearly thereto.

Can you show it from example? — Nine are the square of 3, and 16 of 4; therefore every number between 9 and 16 may be considered the approximate square of 3 and a fraction.

What would their factors be called? — A number which, multiplied once into itself, will produce any other required number, is called its square root.

What is the unit of square measure? — The unit of square measure is the foot square, or 144 square inches.

Produced, or otherwise? — Whatever surface or solid be taken as the unit of square and cubic measures, respectively, it must be assumed; that is, taken arbitrarily; for no linear measure can, by a single involution, produce a square, nor, by a second involution, a solid, of two or three dimensions, each equal to itself.

How does this appear? — Of factors, one only, or the true multiplicand, can signify things; any other, whencesoever derived, can denote, in operation, turns only, or parts of turns; therefore no number of linear feet, inches, or what else, of customary admeasurement, can swell its immeasurably narrow

lines to a figure of two or three dimensions, each equal to itself, by a second or third involution.

To what uses is this measure applied? — Square measure is of use for taking the dimensions of land, of plank, and of glass; for the estimation also of superficial work, as plastering, paving, roofing, house-painting, &c., by numbers called duodecimals.

What is a prime? — A prime is a denomination in duodecimals for the largest division of the foot square.

Can you recite the table?

SQUARE MEASURE.

1′ prime	= a line of 12 sq. inches.	(Marked) ⏋.
144⏋ square inches	= 1 foot square.	⏋f.
9 square feet	= 1 yard square.	⏋y.
30¼ square yards	= 1 rod square.	⏋r.
40 square rods	= 1 rood.	*rood.*
4 roods	= 1 acre.	*ac.*
160 acres	= ¼ section U. S. land.	
640 acres	= 1 mile square.	⏋m.

The rood and acre are square measures only.

SOLIDS.

What is a cube? — A cube is a square solid of three equal dimensions, length, breadth, and height, or depth.

What is the cube of a number? — The cube of a number is its third power.

Why so named? — The square of a number represents a plane figure of two equal dimensions; if a series of planes be continued under or over those two, to the height or depth of a third equal dimension, the whole figure must consist of as many smaller equal solids, as there are units produced by a third equal factor; the number therefore, truly representing three equal dimensions, is fitly called by the name of the figure.

What are the equal factors called? — The number producing a given power, by a second multiplication into itself, is called its cube root.

How is the number of these solids demonstrated? — Let the cube be 8; its cube root is 2; the number of small squares in its plane is four; then, if the units represent square inches, and the cube, raised to the height of 2 inches, be halved, each half will contain 4 solids, of an inch square every way.

Suppose the root be a fraction? — If the root be a fraction, the solid taken will be a portion of that supposed to be represented by unity.

How are the contents of a body of three dimensions found? — *To find the cubic dimensions of any four-sided solid or hollow, multiply length, breadth, and height, or depth, one into another.*

What will this produce? — The product will represent a cube the contents of which are equal to those of the figure under consideration.

How is this demonstrated? — Every number may be considered a cube, for its root can be found, either perfect or approximate.

What is the unit of this measure? — The unit of cubic measure is the solid foot of 1728 cubic inches.

To what use is this measure applied? — Cubic measure is used in estimating the contents of solids and of hollows, such as instrumental measures of capacity, reservoirs, timber, stone, freight.

Can you recite the table?

CUBIC MEASURE.

1728*ci.* cubic inches	= 1 cubic foot. (Marked)	*cf.*
277¼ cubic inches	= 1 standard gallon.	
27 cubic feet	= 1 cubic yard.	*cy.*
40 cubic feet	= 1 ton of freight.	
95 cubic feet	= 1 ton of shipping.	
16 cubic feet, or 4*ft.* in length and breadth, and 1*ft.* in height,	= 1 cord foot.	
128 cubic feet, or 8*ft.* in length, 4*ft.* in breadth and height,	= 1 cord of wood.	

DRAPERIES.

What measure have the manufacturers? — The manufacturers' is called cloth measure.

Of what dimensions? — Cloth measure is a linear measure.

Does not the value of cloth very much depend on its width? — Cloths are made of varying lengths to one and the same width; and their price is determined by lengths of some particular known width.

Is there any standard of cloth measure? — The standard of English cloth measure seems anciently to have been the ell, or length of the arm, estimated equal to a yard and a quarter.

Is the ell still used as a measure of cloth? — The ell is

probably still used by merchants in their transactions abroad; but in the United States we hear only of yards and quarters.

Can you recite the table?

CLOTH MEASURE.

$2\frac{1}{4}$ inches	= 1 nail.	(Marked) *n.*
4 nails	= 1 quarter.	*qr.*
3 quarters	= 1 ell Flemish.	
4 quarters, or 36 in.	= 1 yard.	*yd.*
5 quarters	= 1 ell English.	*El.*

Time.

Its nature.

What is time? — Time is the measure of existence.

In what does measure essentially consist? — Measure, in its largest acceptation, is the comparison of one thing with another, in respect of some property common to both.

Then how is existence measured? — Earthly existence is measured by heavenly existence, or that of the heavenly bodies.

How is their existence measured? — The measure of the existence of the heavenly bodies is the number of their own revolutions.

What may be concluded from this? — Hence it is evident, that one existence is the measure of another, where a thing does not supply the measure of its own existence, by some constantly and equally acting power.

Has the measure then no substantiality of its own? — Time is not an instrumental measure, like the wood and the brass we apply in admeasurement; it is a numbering of revolutions, or the number itself thence derived; which, like all other numbers, has no existence independent of the object numbered.

Does not language seem to imply the contrary? — Common language, by concealing the thing reckoned, makes time to pass for something in itself.

Have you an instance? — The years which we reckon are but so many apparent revolutions of the sun in the ecliptic.

What is eternity? — Eternity is existence not measurable, having neither beginning nor end.

What does time imply? — Time implies an eternity; for limited existence requires some being of existence unlimited, whence it might derive its origin; it must otherwise be supposed to begin from nothing, which is an absurdity.

Might not one limited existence be derived from another through eternity?—This manner of speaking attributes to eternity what we erroneously attribute to time, a something of its own; eternity is the existence of an eternal being, and to allow the one is to confess the other.

Yet might not limited existences succeed one another for ever?—Succession implies beginning, and for ever is without end; whatever begins may end, for it has no self-existence, and every thing has a beginning; therefore, contrary to the supposition, *all* things may come to an end; and being liable all to come to an end, they have, all and every, no self-existence, and therefore, without the intervention of a self-existent being, never could exist at all.

Is it possible that matter should be the eternal being?—Matter is an unintelligent, inanimate thing, according to all that we know of it, and to any definition we can give of it; and if it be pretended that matter is any thing higher, then it is no longer matter that is spoken of, but spirit, cloaked with the name of matter, sometimes from illusion, commonly for the purpose of imposition.

What is the inference?—Since the world is replete with animate and intelligent beings, all of limited existence, the eternal being is one of power and intelligence without bounds; for no cause can be inferior to its effects; and we have shown, that, without an eternal, self-existent cause, effects of no kind could ever have existed.

How does the evidence of facts bear on this question?—Whole races of animals have for ever perished, as the researches of geologists make known to us.

How does the existence of man bear on it?—If man be not a created being, his race must have been without a beginning; but all we know of him from history and philosophy proves him to have had a beginning, and that only a few thousand years ago.

What are the facts?—No human remains have been discovered of any thing like the antiquity of other animal remains; Lucretius, a celebrated poet and atheist of old, has remarked the recent origin of our race from the scanty dates of history, and the recent invention of arts and sciences; and with this supposition agrees every memorial of man, monumental and literary, into which modern research has penetrated.

"Yet more:
If heaven and earth from everlasting be,
And natal day, and origin, knew none;

Whence comes it that beyond the Theban war,
And flaming obsequies of fated Troy,
None other tale nor poet should have sung,
Nor fame, on her undying records, traced
The deeds of heroes and the rule of kings?
'T is new! the whole is new, and every part!
This world not long ago began to be!
Hence, still advancing art a polish takes,
And new inventions teem. The vessel's rounding side
Majestic rises high above the wave;
And minstrels now give life to strains divine;
And only now this very scheme of things,
This nature, so mysterious, is explored;
And now the man discovered, even I,
Who first, in Rome's proud tongue, the mystery reveal."
De Rerum Naturâ. Lib. V. ver. 325—338.

Divisions.

What are the natural divisions of time?—Time is naturally divided into days, months, and years.

What is the first mentioned?—A day is one revolution of the earth round its axis; this includes the period of day and night.

Has the day any other divisions?—Beside the natural divisions of sunrise, noon, and sunsetting, the day is artificially divided into 24 hours.

How are these divisions known?—The hours are known and numbered by means of dials and clocks.

What is true clock time?—True clock time is the length of the astronomical day, such as it would be were the orbit of the earth perfectly circular.

Does the time shown by the sun differ from true time?—Solar time, which is observed on the dial, is always before or after true clock time, except on four days of the year: these are 15th April, 15th June, 31st August, 24th December.

To what is this owing?—It is owing to the unequal motion of the earth in an orbit somewhat oval; hence the natural day is, for the most part, either shortened or prolonged.

And not so with the clock?—The machinery of a good clock moves equably at all times, or may be regulated to an equable motion.

How is true time ascertained, for their regulation?—A sundial, made to the latitude, and truly set, shows the time of the solar day; to this time must be added the equation contained in a correct almanac, when the sun is noted as slow; and deducted therefrom, when the sun is noted as fast.

What is the greatest difference arising from this cause?—The greatest difference occurs about the beginning of Novem-

ber, the sun being then more than 16 minutes faster than the clock.

What is a month? — Strictly speaking, a month is the period of the moon's revolution round the earth; as generally understood and limited, it is a twelfth portion of the year, commencing on a day fixed.

How many lunar revolutions are there in the year? — The moon makes something more than twelve revolutions in the year.

Why then are the larger divisions of the year named from them? — The year, first reckoned by moons in consequence of the remarkable changes of our satellite, would continue so to be reckoned, till the variance of the seasons from the expected time of their arrival compelled mankind to adopt some other mode of reckoning, although the appellation *month* might still be used to divide the year into twelve large periods, no longer regulated by the moon.

What is the common distinction of months? — Months are distinguished into lunar and calendar, the lunar consisting of about 29½ days, erroneously computed even in ancient law, at 28 days; the calendar, either of 30 or 31 days; with one exception.

What is that exception? — February, which, in common years, has 28 days, in leap year, by the intercalation of one, has 29 days.

How many days did the lunar month give to the year? — Twelve lunar months give 354 days to the year.

What is now considered a year? — A year is one revolution of the earth in its orbit round the sun.

When does it begin? — The Julian year, as it is called, begins on 1st January.

At any astronomical period? — At no astronomical period; for if the year be properly reckoned from the winter solstice, which the ancients called the birth of the sun, it ought to begin on 22d December, not ten days later.

By whom was this reckoning established? — When Sosigenes, an astronomer of Alexandria, had reformed the calendar, by order of Julius Cæsar, this emperor, to conciliate the prejudice of the Romans, accustomed to the lunar year, ordained, that the first reformed year should begin on the day of the new moon, immediately after the day of the winter solstice.

What was the effect? — The computed beginning of the year was thus deferred to the eleventh day after its real commencement, as calculated from the solstice.

Modern reformation of the calendar.

Has the calendar been since reformed ? — The calendar was again reformed by pope Gregory XIII, who left the beginning of the year where he found it, but computed its length at 365 days, 5 hours, 49*m.* 12*s.* Still more accurate observation has shown the minutes to be some seconds less than 49.

What is the origin of leap year ? — As the year consists of several hours in excess of complete days, it becomes necessary, with a view to preserve the computation of the seasons in their natural order, that, when the fractions of a day amount to a unit, there shall be added to the reckoning of the year in which the entire unit arises, one entire day ; this, short of a few minutes, is the case every fourth year, hence considered to leap over a day, because affording of itself less than six hours ; and thence called *leap year.*

How is the deficiency of minutes compensated ? — Were the excess of time six complete hours above the days in every year, the intercalation of a day in leap year would render the computation perfect ; but, to compensate the supposed deficiency of between ten and eleven minutes, the same pope Gregory ordained, that, as in single years, so in centuries, every first, second, and third hundredth year should be reckoned common, and every four hundredth, leap year.

Does this make the correction perfect ? — This correction is so nearly perfect, as to give an excess of one day only in three thousand six hundred years.

Could it be made still more perfect ? — The exactitude with which such calculations are made since the days of Galileo, Kepler, and Newton, shows, that every 128th year ought to be excluded from the leap years ; hence would arise the excess of a day in not fewer than 176432 years.

Is this a sufficient reason for changing the rule ? — The Gregorian rule has the advantage of great simplicity and facility in application ; the redundancy given by it will easily be corrected should the period ever arrive ; it would therefore seem best to retain the rule.

What is meant by the difference of style ? — Old style is the Julian year ; new style is pope Gregory's computation, by which eleven days are added to every reckoning in old style.

Do you recollect an instance ? — The pilgrims landed at Plymouth, Massachusetts, on the 11th December, 1620, O. S. answering to the 22d December, N. S.

This being the very day of the solstice, may well seem fitter to be observed as new year's day, than 1st January, the era of a pagan *festival,* and no astronomical period.

Is the old style anywhere used to this day? — O. S. is even now used in Russia, and probably by all who profess the Greek faith.

When was it changed in Britain? — The introduction of N. S. into Great Britain was on the 3d September, 1752, Old Style; by statute then reckoned the 14th September, N. S.

Was any other change of computation then made? — The computed beginning of the year also was then changed from spring to mid-winter.

On what day did it formerly begin? — The year formerly began with us on the 25th March, O. S.

What difference does this make in computation? — The consequence of this anticipation of the year is, that, in ancient writings in our tongue, all dates during the months of January, February, and to the 25th March, must have one year added to their computation, if they are to be made to agree with our present reckoning.

What is the present rule for finding leap year?

MAXIMS IN FINDING LEAP YEAR.

Every year of our Lord measured by 4, and not centennial, is a leap year.

Every fourth centennial year of our Lord is also leap year; all other centennial years are common.

Weeks, &c.

What is the meaning of hebdomadal? — *Hebdomadal* is from a Greek word, and signifies, by the week.

What is a week? — Weeks are an artificial division of time into periods of seven days, arising from the command of the Almighty to observe the seventh day as a day of holy rest.

Why is the division still maintained? — The maintenance of the distinction of time by weeks rests, in part, on the great utility and convenience of it, but chiefly on the divine institution; which being coeval with the creation of man, and in the reason assigned for it alike applicable to all men, is therefore obligatory on all, and for ever.

Can you now recite the table?

MEASURE OF TIME.

60'' seconds	= 1 minute. (Marked)	'
60 minutes or 15° on the dial	= 1 hour.	h.
24 hours	= 1 astronomical day.	d.
7 days	= 1 week.	w.
29½ days	= 1 lunar month nearly.	(Exactly, 29d. 12h. 42' 20''.
30 days	= 1 banking month.	m.
365d. 5h. 48' 48''	= 1 solar year.	yr.
366 days	= 1 leap year.	
52 weeks	= 1 hebdomadal year.	
12 calendar months	= 1 civil year.	
100 years	= 1 century.	

Divisions smaller than the second are now reckoned decimally.

The intercalation of a day is always a question in estimating the days of any particular year or years.

The number of days in a calendar month is easily determined by recollecting the following distich:

> Thirty days hath September,
> April, June, and November.

By banking month is intended the period of days, of indefinite commencement, reckoned to a month among bankers and merchants.

Motion.

What is meant by the place of a body? — *The place of a body is its position in relation to other bodies; and if considered as of a part, place is position with relation to other parts of the same body.*

Can you exemplify this? — If asked where is Boston, I answer by declaring its relation to Massachusetts; namely, that it is therein; if asked, where is the hand, I answer, at the extremity of the arm.

What is motion? — *Motion is active change of place.*

Why active? — Because the place of a body being a relation only, bodies not moved are the subjects of *passive* change of place.

Have you an instance? — The sun is a fixed body, yet is continually changing its place with respect to the earth or parts of it, in consequence of the earth's motion.

In what sense is action attributable to matter? — Action is attributable to matter instrumentally only; as one wheel acts on another, the element on the first wheel; but, primarily, an *immaterial* power on the whole.

Though change of place, in some manner of expressing it, has been for ages the definition of motion, its correctness is still a subject of dispute. To the author, the difficulty seems to arise from not distinguishing between a change acted, and a change endured; for relative and absolute by no means express the true distinction, all change of place being but a change of relation. Bishop Gleig, in discussing this question (Ency. Brit. 3d ed. art. *Motion*), has a long and misty argument to show that motion is a simple idea, and therefore incapable of definition. From this it is evident, that he is obnoxious to the charge of having mistaken the point in debate; since no one understands by motion the power of producing it, but the *product* itself; not an agent, but an action; and an action is definable, or describable, by its effects. It may be objected to the term *active* in our definition of motion, that it implies motion; perhaps it does; but the true meaning of action is, *the exercise of a power*, or power in exercise; and its opposition to *passive* renders the sense in which it is used manifest; *energetic* would perhaps be the true qualifying term; but from the meaning ordinarily given to it, the phraseology would appear ridiculous.

What is the motion of the heavens? — With the exception of comets, the motion of the heavenly bodies, real or apparent, is circular, or nearly so.

What is a circle? — A circle is a line every point of which is equally distant from a common centre.

What is the name given to that line, and whence derived? — The line which constitutes a circle is called its circumference, from a Latin word, signifying *to carry round.*

How is the circle divided? — Circumferences are divided into arcs, or portions taken at will, and into degrees, of an extent defined.

What then is a degree? — A degree is the standard division of any circle, and the 360th part of it.

Why of any circle? — Because the centre of every circle is a point; around the same point may be formed circles of any dimension whatever; and it is manifest, that any two lines proceeding from that centre will cut off an equal proportion of each circumference.

What kind of a measure then is it? — A degree is a proportional measure of the circle; but the arc cut off by it varies in absolute length with the greatness of the circumference.

Have you an example? — A degree on the equator is 69·07 miles in length; toward the pole, the circle itself becomes immeasurably small.

What are the divisions of the degree? — The degree is divided into sixty parts, or minutes, often improperly called miles.

What is to be understood by minutes? — Minute signifies only a minúte portion; and is therefore applied to extension as well as to time.

How is extension applicable to motion ? — Motion is measured by the extent passed over, and the time consumed.

How is the time itself ascertained ? — The measure called time is obtained by comparing the revolution of one body with that of another deemed invariable; as the revolution of a clock index with the apparent revolution of the sun.

How are these made to coincide ? — For every fifteen degrees of the heavens, seemingly passed over by the sun, and indicated by the shadow on the sun dial, the hour hand of most time-keepers passes over thirty degrees of the clock dial.

Why more than fifteen degrees ? — When the hour hand of a clock makes a complete circuit of its face twice in twenty-four hours, the degrees passed over must be double the number of the celestial degrees; but in parts of Italy the hours are reckoned from one to twenty-four, and the clock face divided accordingly.

What is the nature of this process of comparison between one motion and another ? — The clock simply serves to divide and to register the motion of the sun, and to note its inequalities.

What produces the apparent motion of the sun ? — The apparent motion of the sun is occasioned by the real motion of the earth; as the apparent motion of the shore is produced by the real motion of a ship.

What gives motion to the earth ? — The earth is moved round the sun by the operation of two forces, or powers, called the centripetal, which tends toward the centre of the sun, and the centrifugal, which propels in a contrary direction; these two balanced powers give the earth a rotatory motion, and carry it round the sun.

What familiar action may give an idea of this ? — We may perhaps conceive of it, by a comparison with the whirling of a stone in a sling round the hand.

Power.

What is power ? — *Power is an attribute of mind, subjecting all matter to its control, the cause therefore of all motion.*

How is this demonstrated ? — In every work of man we know that mind is the sole contriver, director, and, by the intervention of instruments, the sole operator; for the limbs of a man are his instruments, and his mind moves his limbs.

What do you say to the case of the paralytic ? — The case of the paralytic is the same with that of a man who has lost his limbs; human minds act by the instrumentality of human bodies; when these become imperfect in any material respect, *no matter what*, the mind has lost its first instrument.

How does the argument apply where an element is interposed? — When fire, water, wind, are interposed, the mind of man first designs, then, by means of the hand, arranges, his machines, so as to receive the action of the element upon them; and the action itself is from the divine mind.

Have the elements then no power of their own? — Material things can have no power of their own, for they are inanimate; that which is lifeless, the elements not excepted, is disposed of at our pleasure, according to the extent of human power; therefore all inanimate things are irresistibly disposed of by the divine power.

Can we see this amazing power? — Our own minds are not seen, though producing effects most visible.

Do you recollect any other familiar illustration of this subject? — If the attention of an uninstructed person were called to the vibrations of a pendulum fitted within a frame which concealed all the connecting parts, he might imagine the ball to be something self-moving; as ignorant persons may suppose of the sun, moon, and planets; and as atheists would have us believe.

What is your conclusion from the whole? — *Mind is the only power in the universe; all the great operations in nature are from the action of the divine mind; this action we continually behold in its effects, produced according to the laws of the Divinity's own prescribing to himself, laws invariable except in the case of his miraculous interposition.*

Angles.

What is an angle? — An angle is the inclination, to each other, of two lines that meet at a point.

How are they measured? — Angles are measured by the arcs they subtend.

Of what circle? — Of any circle, the inclination being alike to all circles; since from any point all circles whatever may be described.

Can you recite the usual divisions of the circle?

CIRCULAR MEASURE.

60″ seconds	=	1 minute.	(Marked) ′
60 minutes	=	1 degree.	°
30 degrees	=	1 sign of the zodiac.	
90 degrees	=	1 quadrant.	
180 degrees	=	1 semicircle.	
360 degrees	=	1 circle.	

Instead of a subdivision into thirds, the second is now divided decimally.

What are the papermaker's divisions?

PAPER.

24 sheets = 1 quire.
20 quires = 1 ream.
2 reams = 1 bundle.
2 bundles = 1 bale.

Are you acquainted with any general divisions?

GENERAL DIVISIONS.

20 things = 1 score.
12 things = 1 dozen.
12 dozen = 1 gross.

Reduction.

What is the meaning of the word *reduce*? — To reduce is commonly understood as meaning to lower; but it also signifies, according to its etymology, to bring back, to restore.

What is the effect of reducing a dollar to cents? — A dollar reduced to cents is lowered in denomination, while its value remains unaltered.

Suppose the cents restored to the form of dollars? — Cents reduced, or brought back, to dollars are raised in denomination, without change of value.

Then can you define the reduction of composites? — ***Reduction of composites is the lowering of numbers of a higher denomination, called reduction downward; and the raising of numbers of a lower denomination, called reduction upward; in either case without change of value.***

What is the term peculiar to this rule, and whence derived? — ***Reducend*** is a Latin word abbreviated, signifying something to be reduced.

Suppose it were pence to shillings? — To reduce pence to shillings, I would divide the pence by the number contained in a shilling; for by division lower values are raised to higher.

Suppose the reverse were required? — To reduce shillings to pence, I would reverse the operation, and multiply the shillings by the number of pence contained in a shilling; for thus the number would be increased as the value was lowered.

How would you prove the correctness of these reductions? — The two kinds of reduction must prove each other; for the numbers added by the one are subtracted by the other.

Can you now state the rule?

RULE OF COMPOSITE REDUCTION.

To lower composites, multiply the highest denomination given into as many of the next lower as are contained in a unit of the higher; to the product add any of the lower contained in the reducend, and so proceed to the lowest.

To raise composites, divide the reducend by as many of the same as are contained in a unit of the next higher denomination; note any remainder as composite, and so proceed to the highest.

Proof of reduction upward by reduction downward, and the reverse; but, should a fraction occur in the factors, the proof, on reduction upward, may differ in denomination, though not in value, from the reducend.

Fractions of composites.

How are fractions of composites reduced in denomination? — Fractions of composites are reduced precisely as their units; to lower denominations by reduction downward; to higher, by reduction upward.

With what difference of manner? — *Fractions of composites are lowered in denomination by multiplying the numerator into as many of the lower as are contained in a unit of the higher; and raised in denomination, by multiplying the denominator into as many of the lower as are contained in a unit of the higher.*

With what effect? — Thus they are proportionably increased or diminished in the amount newly denominated, while their absolute value continues the same.

How are fractions reduced to the composite form? — Fractions are reduced to the composite form, first, by reduction downward (there being no reduction upward of such fractions); then, by reduction of the improper fraction produced.

APPLICATION.

1. How many inches are there in a degree of a great circle?

Downward.

69·07*ml.*
8*fu.* = 1*ml.*

552·56
40*r.* = 1*fu.*

22102·40
5½*yd.* = 1*r.*

11051·2
110512·0

121563·2
3*ft.* = 1*yd.*

364689·6
12*i.* = 1*ft.*

4376275·2*i.* answer.

Upward.

12*i.* : 4376275·2*i.*

3*ft.* : 364689·6

5½*yd.* : 121563·2
2 2

11 : 243126·4

4,0*r.* : 2210,2·4

8*fu.* : 552·56

69·07*ml.* proof.

In the proof we simplify the mixed divisor, 5½*yd.* and increase the dividend. Having begun the reduction with a mixed decimal factor, we have divided decimally in the proof, instead of noting remainders as composite.

2. How many lines in 73 miles, 7 furlongs, 39½ rods, 2 feet.

Downward.

73*ml.* 7*fu.* 39½*r.* 2*ft.*
8*fu.* = 1*ml.*

591
40*r.* = 1*fu.*

23679½
5½*yd.* = 1*r.*

11839¾
118397½

130237¼
3*ft.* = 1*yd.*

390713¾
120' = 1*ft.*

90
4688556

46885650' answer.

Upward.

12,0' : 4688565.0'

3*ft.* : 390713 9*i.*

5½*yd.* : 130237 2*ft.*
2 2

11 : 260474

4,0*r.* : 2367,9 + $\frac{5}{2}$ $\frac{5}{2}$*yd.* = 2½*yd.*

8*fu.* : 591 39*r.*

proof, 73*ml.* 7*fu.* 39*r.* 2½*yd.* 2*ft.* 9*i.*

½*r.* = 2¾*yd.* = 2½*yd.* 9*i.*

In this example we have almost every peculiarity that can occur in the reduction of composites; numbers of lower denominations to be added from the reducend, usually so added to the first partial products, and the frequent occasion of error to the young arithmetician; in the proof, a mixed divisor, division of a remainder, and change of denomination, from the circumstance of the factors containing fractions. Notice first in the upward reduction, that remainders are left, according to the rule, as composite from the denomination divided. Thus, 9, left on the first division, are noted as 9 inches; because the decimation of the dividend had raised it from lines to inches. Five, left by the divisor, 11, are so many halves, and noted accordingly, because the dividend had been doubled; and the denominations, $9i.$ and $2\frac{1}{2}yd.$, that do not appear in the reducend, are compared with the deficiency of $\frac{1}{2}r.$ in the proof, and found exactly to compensate it. In point of correctness, fractions not belonging to a table ought to be reduced to their composites, and be made to appear in the reducend as they at length appear in the proof; and this method may be taken in the examples that follow; but, as the contrary case does sometimes occur, and occasions serious difficulty to the learner, its exemplification must be an advantage.

3. What fraction of a pound is $\frac{1}{2}$ a shilling?

By reduction upward, $20 :\cdot \frac{1}{2} = £\frac{1}{40}$, answer.

4. What is the composite value of $£\frac{1}{40}$?

By reduction downward, $£\frac{1}{40} \times 20 = \frac{1}{2}s.$ $\frac{1}{2}s. \times 12 = \frac{12}{2}d. = 6d.$, ans.

The reduction of composites is in reality a proportional operation; for the reducend is money of one name requiring estimate in money of a different name; but as either the article estimated, or the estimate given, is always a unit, and as a single reduction often combines a series of proportions, it has been found not only practicable, but highly convenient, to make a distinct rule of this branch of proportion. To exemplify our remark, let us suppose £50 requiring estimation in pence; the ratios are: $\frac{50}{1} \times 20 = 1000s.$; $\frac{1000}{1} \times 12d. = 12000d.$ Reversed; $\frac{12000}{12} \times 1s. = 1000s.$; $\frac{1000}{20} \times £1 = £50.$

Examples to be wrought, proved, and recited.

1. In six hundred thousand five hundred seventy-three dollars, how many mills?

2. In nine hundred ninety-five eagles, how many dollars, cents, and mills?

3. In £486 17*s.*, how many shillings, pence, and farthings?

4. In six thousand fifteen sovereigns, how many pounds, shillings, pence, and farthings?

5. In three hundred and four guineas, how many pounds, shillings, pence, and farthings?

6. In three hundred seventy-one thousand nine hundred three farthings, how many guineas?

7. In £298 3*s*. 4$\frac{1}{2}$*d*., how many shillings, pence, and farthings?

8. In seven hundred fourteen thousand fifty-nine pence, how many shillings and pounds?

9. In £1145 19*s*. 11$\frac{3}{4}$*d*., how many guineas?

10. In $\mathcal{F}$ 16℔ 7 *oz*. 3 *dw*. 7 *gr*., how many grains?

11. In fifty-five thousand five hundred sixty-one grains, troy, how many pennyweights, ounces, and pounds?

12. In 7$\frac{1}{2}$℔ apothecaries' weight, how many scruples?

13. In Q76$\frac{1}{4}$, how many drachms avoirdupois?

14. In four hundred seven million fifteen hundred sixty-two pounds avoirdupois, how many quintals and tons?

15. In *T* 55 16$\frac{1}{2}$ Q 14℔, how many drachms?

16. In five hundred seventy-eight million thirteen thousand six hundred ninety-eight drachms, how many hundred weight?

17. In *G* 76 *qr*. 1 *ga*., how many pints?

18. In one hundred forty-eight thousand three hundred four pints, how many grain quarters?

19. In *L* 63$\frac{3}{4}$ *ga*., how many gills?

20. In *L* 3$\frac{1}{2}$ *hhd*., how many pints?

21. In *E* 2° 1 *ml*. 3$\frac{1}{2}$ *yd*., how many inches?

22. In nine billion fifty-four million nine thousand three hundred twelve inches, how many miles, yards, &c.?

23. In 3 *lg*. 1 *ml*. 6 *fu*. 27 *r*. 2 *ft*., how many lines?

24. In a section of land, how many square inches?

25. In six million three hundred eighty thousand nine hundred sixteen square inches, how many acres, square feet, &c.?

26. In 27 *ac*. 15$\frac{1}{2}$⌉*yd*. 7⌉*ft*., how many square inches?

27. In a ton of freight, how many cubic inches?

28. In 5 *El*., how many nails and inches?

29. In 17 *yr*., how many months, weeks, days, hours, and minutes?

30. In three million nine hundred four thousand six hundred seven minutes, how many hours, days, weeks, months, and years?

31. In 10 *yr*. 287 *da*. 13$\frac{1}{2}$ *h*. 48″, how many seconds?

32. What fractions of the pound sterling are $\frac{3}{4}$*d*.? $\frac{5}{6}$*d*.? $\frac{7}{12}$*s*.? $\frac{13}{14}$*s*.?

33. What fractions of the pound troy are $\frac{7}{10}$*gr*.? $\frac{9}{11}$*dw*.? $\frac{3}{7}$*oz*.?

34. What are the composite values of \$$\frac{3}{9}$? \$$\frac{5}{7}$? \$$\frac{7}{8}$? £$\frac{8}{17}$? £$\frac{25}{109}$? $\frac{2}{19}$*s*.? $\frac{6}{13}$*s*.? $\frac{55}{104}$?

35. What are the composite values of $\mathcal{F}$ $\frac{7}{11}$? Q$\frac{19}{58}$? *G* $\frac{3}{5}$ *bu*.? *L* $\frac{11}{63}$ *p*.? *L* $\frac{7}{37}$ *hhd*.? $\frac{23}{90}$ *ml*.? $\frac{39}{115}$⌉*ml*.?

36. What are the composite values of $\frac{12}{17}$ *cy*.? *El*.$\frac{6}{7}$? $\frac{120}{219}$ *yr*.? $\frac{39}{51}$ *mo*.? $\frac{17}{60}$ *da*.? $\frac{73}{125}$°? $\frac{11}{12}$ *rm*.?

Composites to decimals.

What are composites? — Composites are units distributable into parts, or smaller values, having invariable denominators and peculiar names.

How are common fractions changed to decimals? — Common are reduced to decimal fractions by the actual division of numerator by denominator.

What kind of parts are composites distributable into? — The parts of composites are common fractions; their denominators, though not expressed, being always supposed.

Then what decimal of a penny is one farthing? — A farthing is twenty-five hundredths of a penny; for 25 are a fourth part of 100, and by decimal division this is the quotient obtained. $\frac{1}{4}d. = \cdot25d.$

How would you reduce one penny farthing to decimals of a shilling? — A penny farthing is a complex fraction of a shilling, for the denominator of pence is 12; this I might simplify, and then divide numerator by denominator.

$$1\tfrac{1}{4}d. = \frac{1\frac{1}{4}}{12} = \frac{5}{48} = \cdot1041\dot{6}s.$$

Can decimals be terms of a common fraction? — Decimals form complex fractions needing no simplification.

Does this afford any means of retaining tabular denominators in cases like the present? — In the reduction of different composites, the lowest may first be reduced to a decimal, then a higher prefixed to it, without increasing the denominator.

$$\tfrac{1}{4}d. = \cdot25d.;\ \text{then, } 1\tfrac{1}{4}d. = \frac{1\cdot25}{12} = \cdot1041\dot{6}s.$$

Could these operations be formed into a more regular series? — By setting the composites to be reduced, one under another, beginning with the lowest, and figuring the decimal quotient of a lower value on the right of a higher, the divisions will successively be made in exact order, according to reduction upward.

$$\begin{array}{r}
4 : 1d. \\
12d. : 1\cdot25d. \\
\hline
\cdot1041\dot{6}s.
\end{array}$$

What fractional value of a pound is one farthing? — In a pound sterling there are 960 farthings; because 4 farthings make 1 penny, 48 therefore make one shilling; and twenty times 48 are 960; consequently a farthing is £$\frac{1}{960}$.

$$12d. : \tfrac{1}{4}d. = \tfrac{1}{48}s.$$
$$20s. : \tfrac{1}{48}s. = £\tfrac{1}{960}.$$

What is the decimal equivalent to this fraction? — The decimal value of £$\frac{1}{960}$ appears, on division and reduction, to be one thousandth and one twenty-fourth of a pound sterling; as stated also in the table. $96{,}0 : 1\cdot00 = \cdot001\frac{4}{96} = \cdot001\frac{1}{24}.$

What decimal of a pound is one penny? — A penny must be four times as much; or £·004$\frac{1}{6}$. ·001$\frac{1}{24}$×4 = ·004$\frac{1}{6}$.

What decimal of a pound are two shillings? — Twenty shillings making a pound sterling, 2*s*. are manifestly £·1.

$\frac{2}{20}$ = ·1.

A single shilling? — One shilling is the half, or five tenths, of two shillings; it is therefore five hundredths of a pound.

$\frac{1}{20}$ = ·05.

Decimals to composites.

How may such decimals be restored to the compound form? — Decimals not in the lowest denomination are reduced to composites by reduction downward.

Why downward? — A decimal of denomination cannot be raised to a composite of higher, for it is itself but a fraction of one lower; and because it is a fraction, it cannot be reduced to a composite form of equal denomination.

How will the composites appear on such a reduction? — The integers, as produced, are the composites required.

Can you exemplify this? — Five hundredths of a pound sterling, multiplied by 20, produce the composite 1*s*.

$$\begin{array}{r} £\cdot05 \\ 20 \\ \hline 1\cdot00s. \\ \hline\hline \end{array}$$

Have you learned the rule?

RULE FOR THE INTERCHANGE OF COMPOSITE AND DECIMAL FORMS.

To reduce composites to decimals, arrange the composite numbers separately, one under another, beginning with the lowest denomination. By reduction upward, raise that of the lowest denomination to decimals of the next higher, figuring the quotient on the right also of the next higher; and so proceed, raising every successive mixed number to decimals of a higher denomination, up to the highest.

Decimals of the pound sterling are easily found on inspection, by mental computation only, or recollection of the table.

Decimals of denomination are reduced to composites of succeeding lower denominations by reduction downward; the integers, as produced, being the composites required.

Proof. Reduction to one form is proved by restoration to the other.

APPLICATION.

1. What decimal of a pound sterling is equivalent to the composite sum, 7*s*. 11¼*d*.?

By reduction upward.	By inspection.	By inspection.
4 : 1*f*.	7*s*. =£·35	£·3 =6*s*.
12*d*. : 11·25*d*.	6*d*. = ·025	·05 =1
2,0*s*. : 7·9375*s*.	5¼ = ·021$\frac{21}{24}$	·025 = 6*d*.
£ ·396875, ans.		·021$\frac{21}{24}$= 5¼
	7*s*. 11¼*d*.=£·396$\frac{21}{24}$, ans.	
$\frac{21}{24000}$ = ·000875.		£·396$\frac{21}{24}$=7*s*. 11¼*d*. proof.

The work by inspection is thus performed: We say, seven shillings are three tenths and five hundredths of a pound sterling; six pence are twenty-five thousandths; five pence farthing are 21 times the value of a single farthing, or 21 thousandths and $\frac{21}{24}$. We prove the same by restoring the decimal to the composite form; this also by inspection; a method very easily practised, when once understood. In doing this we separate the parts of the decimal found, saying three tenths of a pound are six shillings, five hundredths are one shilling; there now remain 46 thousandths and the fraction; of these, 25 are equal to sixpence; and the 21 thousandths $\frac{21}{24}$ still left are equal to exactly the same number of farthings. That the fraction $\frac{21}{24}$ is correct may be proved by reducing it decimally, as done below the first reduction.

2. What is the composite expression of £·0313?

By reduction downward.	By inspection.	By reduction upward.
£.0313	£·025 =6*d*.	12*d*. : 7·512*d*.
20*s*.	·0063=1½+	
		2,0*s*. : ·626
6260	£·0313=7½*d*. ans.	
12*d*.		£·0313, proof.
7·5120*d*.=7½*d*. ans.		

In performing the work by inspection, we know that £·0063 can be reduced only to six farthings, because every farthing requires more than a single thousandth part of the pound; and what we have to determine is, whether the myriadths be in sufficient number to supply $\frac{1}{24}$ to every thousandth part; in the present case they are more than sufficient; as $\frac{6}{24}$ are equal to ¼, a smaller value than $\frac{3}{10}$.

Examples to be wrought, proved, and recited.

1. What decimals of a pound sterling are equivalent to 15*s*. 6*d*.? 19*s*. 4*d*.? 7*s*. 3½*d*.? 13*s*. 11¾*d*.? 8*s*. 7¼*d*.? 11*s*. 0½*d*.? 14*s*. 11¾*d*.? 17*s*. 6¼*d*.? 18*s*. 9¾*d*.? 19*s*. 11¾*d*.?

2. What is the composite expression of the decimals, £·514? £·683? £·067$\frac{2}{3}$? ·73*s.*? ·591*s.*? £·009$\frac{5}{8}$? £·391? ·961*s.*? ·7*d.*? £·0099? £·178$\frac{3}{8}$?

3. What decimals of a pound troy are equivalent to 7 *oz.* 13 *dwt.* 19 *gr.*? 11 *oz.* 9 *dwt.* 23 *gr.*?

4. What are the composite values of ℥ 16℔? ℥ ·073℔? *G* ·128 *gr.*? *L*·78 *hhd.*? *L* ·097 *p.*? ·06 *fu.*? ·17⌝*r.*? ·565 *cf.*? ·731 *yr.*? ·23 *da.*? ·119°?

5. What decimals of a square yard are equivalent to 7⌝*f.* 123⌝*in.*?

6. What decimals of a cubic yard are equivalent to 13 *cf.* 668 *ci.*?

7. What is the composite expression of ·345 *cf.*? ·87⌝*f.*? ·009 *yr.*? ·183 *yr.*? ·371 *da.*?

8. What decimals of a year are equivalent to 169 *da.* 19 *h.* 6′ 59″? 5 *m.* 3 *du.* 5 *h.*?

9. What decimals of a degree, to 37′ 44·309″?

COMPOSITES TO ALIQUOT PARTS.

How are composites reduced to their lowest fractional terms? — Composites, like all other fractions, are reduced to their lowest terms by a common measure of numerator and denominator.

Can you exemplify it? — Fractionally expressed, the composite, 3*d.* (three pence) is $\frac{3}{12}$ of a shilling, reducible by the common divisor 3, to $\frac{1}{4}$ of a shilling.

What is the meaning of the word, aliquot? — *Aliquot* is a Latin word, signifying, some things, indefinitely; or precisely, some certain things.

What are aliquot parts in arithmetic? — Aliquot parts are fractions of a certain kind, usually defined to be such a part of any number as, taken an exact number of times, will produce that number.

Whether this be a just definition of aliquot parts, as they are used in arithmetic, the author does not presume to decide for others; to himself the definition appears perfectly nugatory.

Can you define them according to their ordinary use in accounts? — *Aliquot parts are fractions in their least possible terms, having therefore a unit for their numerator, and denominators that are tabular, or easy divisors in a single line; as* $\frac{1}{4}$, $\frac{1}{10}$.

What is the advantage derived from their use? — By changing composites of a lower, to aliquot parts of a higher, denomination, reduction and long integral multiplication are avoided,

or double operations and long division in fractional multiplication.

Are not all composites of a lower denomination, fractions of some higher? — Composites of a lower denomination are all fractions of some higher; but having commonly numerators more than unity, or denominators operated with by long division, they can seldom be used as aliquot parts without reduction to less terms.

How are they then applied? — Aliquot parts are used to great advantage as fractional multipliers; for having a unit only in their numerators, the product is at once obtained by an easy division.

How may composites having terms which, entire, are irreducible to aliquot parts, be so divided as to answer this purpose? — If the whole of a composite number be too great for an aliquot part, a portion of it may first be taken; then, the remainder; so, if a unit of a lower be not an aliquot part of the higher denomination, it is possible that a larger number may be.

Have you an example? — An ounce avoirdupois is not a convenient aliquot part of the pound; for its denominator is 16; but 2 ounces are, for they are reducible to $\frac{1}{8}$ of a pound; nor are 5 ounces; but 4 may first be taken, since they are $\frac{1}{4}$ of a pound; and the 5th is $\frac{1}{4}$ of $\frac{1}{4}$ of a pound.

To obtain the product of the latter, how would you proceed? — Having multiplied the first $\frac{1}{4}$ into the multiplicand, its product would be the multiplicand of the second $\frac{1}{4}$; for the 5th ounce is $\frac{1}{16}$ of a pound; by submultiples, therefore, the product of $\frac{1}{4}$ becomes the multiplicand of another $\frac{1}{4}$.

How would you finish? — The two separate products would be added together for the entire product.

Can you exemplify this? — Suppose 5 *oz.* purchased, at $1 the ℔; 4 *oz.* are $\frac{1}{4}$ of a pound; the product of $\frac{1}{4}$ of a dollar is 25 cents; 1 *oz.* is $\frac{1}{4}$ of 4 *oz.*; and $\frac{1}{4}$, multiplied into 25 cents, produces $6\frac{1}{4}$ cents.

4 *oz.* = $\frac{1}{4}$ ℔	$1
	·25
1 *oz.* = $\frac{1}{4}$ of $\frac{1}{4}$	·0625
	$ ·3825

Has the use of aliquot parts any distinguishing name? — The use of aliquot parts is commonly denominated *practice*.

Does practice differ from proportional operation? — Practice involves no distinct principle in arithmetic; it is a method of obtaining and applying convenient fractional factors in the solution of certain questions of proportion; for by proportion alone can estimates of any kind be determined.

What are those certain questions? — Aliquot parts are factors very commonly used in the multiplication of one composite into another, when articles are estimated by the unit; as one pound weight, to a price, whatever it be.

Can aliquot parts always be found? — Aliquot parts cannot always be had, as of an inch in a mile; though parts that may be very small and inconvenient fractions of a higher denomination, are not unfrequently arithmetical aliquot parts of a lower in the same sum; as of an inch in a foot.

When not attainable, what substitute have we? — When aliquot parts cannot be obtained, recourse must be had to reduction of composites or to decimals.

What is the general mode of applying them? — Of two composite factors, if one be in a single denomination, the highest denomination of the other may be used as an integral factor, and its lower denominations be reduced to aliquot parts of the same; or if the highest denomination expressed make a convenient aliquot part of one still higher, it may be so used, and reduction thus be avoided.

Can you exemplify this? — If 19℔ of tea be priced at 7*s.* 7*d.* the ℔, 7*s.* will best be used as an integral multiplier of 19℔; because it is easier to find aliquot parts of 7*d.* in a shilling, than of 7*s.* 7*d.* in a pound sterling; but if the price were 6*s.* 8*d.* the pound weight, it would be most convenient to take $\frac{1}{3}$ of 19℔; for 6*s.* 8*d.* is one third of the pound sterling, the operation with such factors is easy, and the product is obtained in the denomination of pounds sterling, without reduction.

How does it appear that the product would be in pounds? — The product sought is value; therefore, if the factor representative of value, be in a lower denomination, the product will be in the same lower denomination; consequently, if raised to a higher denomination, it will produce the higher.

Can you exemplify this? — If 3℔ be rated at 6*s.* 8*d.*, the product of these factors is shillings; for $3 \times 6s.\ 8d. = 20s.$; to raise this product to the denomination of pounds, we divide by 20; for $\frac{20}{20}s. = £1$; but if we first divide either factor by 20, the product will then also be £1.

Arithmetically how would you bring the factors in such a division to an aliquot part? — The fraction of a pound, $\frac{6s.\ 8d.}{20}$, is reduced to $\frac{1}{3}$, an aliquot part of the pound, by means of a common measure; for the numerator measures the denominator.

What limitation is there in the arrangement of composite factors? — A composite factor in a single denomination usually makes the best multiplicand; if two composite factors are,

each, in more than one denomination, that must be made the multiplicand which is the true multiplicand, or the product will not be in the composites desired.

Can you assign the reason ? — If the multiplicand be of one denomination only, any remainder, arising from fractional multiplication, may be reduced to any lower denomination of the true multiplicand, even should this be the actual multiplier.

How does this appear ? — Such a remainder must still be multiplied into the aliquot part; their product is a fraction in the higher denomination, the remainder itself is a numerator; therefore, to reduce it to a lower denomination, we multiply the numerator into as many of the lower as are contained in a unit of the higher.

Can you exemplify the case ? — If 14 *yd.* of cloth be priced at 5*s.* the yard, we may multiply 14 *yd.* by £$\frac{1}{4}$ sterling; the product will be £3, and there will be 2 yards over; this remainder, multiplied into £$\frac{1}{4}$, produces £$\frac{1}{2}$; and £$\frac{1}{2}\times 20=\frac{20}{2}=10s.$

$$5s.=\tfrac{1}{4}\pounds \;\Big|\; 14\,yd.$$
$$\pounds\, 3\tfrac{2}{4}$$

In the case of several denominations ? — If an actual multiplicand have lower denominations, a remainder from the higher must be reduced to the lower, and added thereto, as the rules of reduction require; but the denominations of the true multiplicand may differ entirely in the division of their parts, from those of the other factor.

What is the inference ? — *When composite factors are, each, of several denominations, the true multiplicand must be made the actual multiplicand.*

Can you exemplify this case ? — If 14 *yd.* 3 *na.* of cloth be priced at 5*s.* 6*d.* per yard, there being 16 nails in a yard, any remaining yards must be multiplied into 16, in order to be added to the 3 nails; but a multiplication into 16 will not reduce a fraction of a pound sterling to shillings.

How would you obtain the answer ? — In the present case, value is sought; therefore 5*s.* 6*d.* are the true multiplicand; this may be multiplied into the 14 *yd.* by submultiples and composite multiplication; and into the 3 nails, by reducing them to aliquot parts of the yard. (See the Application, Example 3.)

Can you now give me the rules ?

RULE TO FIND ALIQUOT PARTS.

Aliquot parts are found on inspection, by taking as many units of lower composites, with or without fractions, as, when reduced to their least terms, will make the largest aliquot part

of the highest denomination given or sought; units still remaining are to be raised to smaller aliquot parts of the same, or reduced to aliquot parts of preceding aliquot parts.

Proof. The sum of the aliquot parts taken is a fraction of the higher denomination, equivalent to the composites raised.

RULE IN THE USE OF ALIQUOT PARTS.

Of two composite factors, either one in a single denomination may be made the multiplicand, and any remainder therefrom may be reduced to lower denominations of the true multiplicand, which is of the same kind with the composites desired in the product; but if both factors are in two or more denominations, the true multiplicand must be made the actual multiplicand, and the process may be conducted by composite multiplication for the higher denomination of the multiplier, and by aliquot parts taken in the same higher, for the lower denominations.

If one factor be in a single denomination, the highest denomination of the other may be multiplied into it, and aliquot parts taken in the same highest, for the lower; or if the highest denomination expressed make a convenient aliquot part of one still higher, it may be reduced thereto, and operated with as such. (For proof, see Proportion.)

APPLICATION.

1. At $6 the cwt. American, what is the price of 2*cw.* 19℔ of sugar?

10℔ $=\frac{1}{10}$ *cw.*	$6	$\frac{1}{10}$ *cw.* $=$ ·1
5 $=\frac{1}{20}$	2*cw.*	$\frac{1}{20}$ $=$ ·05
4 $=\frac{1}{25}$		$\frac{1}{25}$ $=$ ·04
	12	
	0·6	·19*cw.* $=$ 19℔, proof of r.
	·3	[to a. p.
	·24	
	$13·14, ans.	

In the present case, the aliquot parts are all multipliers of the same multiplicand, and are on that account braced together. No difficulty is felt in dividing by 25 in a single line, when the computation is decimal, its contents in 100 being so obvious.

2. At 9*s*. 7½*d*. the yard, what is the price of 31 *yd*. of cloth?

6*d*.=$\frac{1}{2}$*s*.	31 *yd*.	Proof of r. to a. p.
	9*s*.	$\frac{1}{2}$*s*.=6*d*.
	279	$\frac{1}{4}$ of $\frac{1}{2}$*s*.=$\frac{1}{8}$*s*.=1½.
1½*d*.=$\frac{1}{4}$ of $\frac{1}{2}$	15 6	$\frac{5}{8}$*s*.=7½*d*.
	3 10½	

2,0 : 29,8 4½=£14 18*s*. 4½*d*. ans.

We might have taken 1½*d*. in the shilling, as $\frac{1}{8}$*s*.; but by taking it in the preceding aliquot part, as $\frac{1}{4}$ of $\frac{1}{2}$*s*., or 6*d*., the multiplier finds its place adjacent to its proper multiplicand.

3. At 5*s*. 6*d*. the yard, what is the price of 14 *yd*. 3 *na*. of cloth?

2*na*.=$\frac{1}{8}$*yd*.	5*s*. 6*d*.	$\frac{1}{8}$*yd*.=2*na*.
	7×2=14 *yd*.	$\frac{1}{2}$ of $\frac{1}{8}$=$\frac{1}{16}$ =1
	1 18 6	$\frac{3}{16}$*yd*.=3*na*.
	2	[proof of r. to a. p.
	3 17 0	
1*na*.=$\frac{1}{2}$ of $\frac{1}{8}$	8¼	
	4$\frac{1}{8}$	

£3 18 0$\frac{3}{8}$, answer.

4. At £1 7*s*. 6*d*. per yard, what is the price of 56 *yd*. of cloth?

6*s*. 8*d*.=$\frac{1}{3}$£	56 ×1£	$\frac{1}{3}$£=6*s*. 8*d*.
10*d*.=$\frac{1}{8}$ of $\frac{1}{3}$	18 13 4	$\frac{1}{8}$ of $\frac{1}{3}$=$\frac{1}{24}$ = 10
	2 6 8	$\frac{9}{24}$£=7*s*. 6*d*. proof of [r. to a. p.

£77 0 0, ans.

Examples to be wrought and recited.

1. What aliquot parts of a dollar are \$·37½? \$·62½? \$·75? \$·81¼? \$·93¾?
2. What aliquot parts of a pound sterling are 15*s*.? 11*s*.? 10*s*. 6*d*.? 8*s*. 4*d*.? 5*s*. 2*d*.? 6*s*. 9*d*.? 16*s*. 8*d*.?
3. What aliquot parts of a pound avoirdupois are 7*oz*.? 6*oz*. 8*dr*.? 5*oz*. 4*dr*.? 2*oz*. 5*dr*.? 13*oz*.?
4. What aliquot parts of a pipe are 1*hhd*.? 3*brl*.? 94½*ga*.? of a tun are 3*hhd*. 15¾*ga*.? 1*p*. 1*hhd*.?
5. What aliquot parts of a mile are 7½*fu*.? 3*fu*. 15*r*.? of a yard are 1½*ft*.? 2*ft*. 6*i*.? 8*i*.? 5¼*i*.

6. What aliquot parts of a year are 5m? 182½da.? 7m. 20da.? 13wk.? 4wk.? 90da.? 180da.?

These examples are intended for mental solution also, on interrogation: none are given for exercise in the *application* of aliquot parts, because this belongs to proportion, and is of extensive range.

Addition.

How would you add pence to shillings? — Pence fewer than twelve are added to shillings by annexation on the right, with their proper mark of denomination.

How would you add a larger number? — Pence equal to or exceeding 12, I would divide by twelve, carry the quotient to the shillings, and set down any remaining pence, properly denominated, on the right.

Why? — Because the use of composites is to lessen numbers in computation; because the change of a lower denomination to a higher requires a lessening of number in proportion to the increase of value; and because sums thus lessened can no longer remain without confusion, under the denomination whence they are raised.

When is a composite in excess? — A composite is in excess of a lower denomination, when it is equal to, or exceeds, the value of a unit in some higher.

Have you learned the rule?

RULE OF COMPOSITE ADDITION.

Arrange composite addenda of the same denomination one under another, and begin to add up on the right. By reduction upward, raise the total of a lower denomination, when in excess, to the next higher; carry the quotient thereto, set down any remainder under the column added up, and so proceed to the highest.

Proof, by reckoning downward; but this affords no proof of the correct arrangement or reduction of the composites, which can only be shown by their agreement with the tables and the rule of reduction.

APPLICATION.

£763	4s.	2¾d.
28	5	0
192	7	6¼
	19	3
5	0	4¼
349	9	11
£1339	6	3¼

Let the addenda be as figured in the margin; what is their sum? — In this example, the farthings amount to 6, which, divided by 4, give 1½d.; one penny added to the column of pence, makes their sum 27; these, divided by 12, the number of pence in a shilling, give 2s. 3d.; two shillings, added to the column of shillings, make their sum to be 46; which, divided by 20, the number of shillings in a pound, give 2 to be carried to the pounds.

An important part of the following exercises consists in the correct arrangement of the addenda; the only guide to this is the tables; and the only means of assurance, a perfect knowledge of them.

Examples to be wrought and recited.

1. What is the sum total of £15 6*s*. 8*d*. + £7 3½*d*. + 19*s*. 10*d*. + £43 1*s*. 6¼*d*.+9*s*. 5*d*.+17*s*. ?
2. What is the total weight of ꟻ 2℔ 5*oz*. 10*dw*. 17*gr*.+8*oz*. 19*dw*.+5℔ 16*gr*.+23℔ 9*dw*.+17℔ 2*oz*. 1*gr*. ?
3. What is the total weight of 5℥ 7ʒ 1℈ 3*gr*.+1℔ 11℥ 5ʒ 2℈ 19*gr*.+3℔ 8℥ 1ʒ 1℈ 5*gr*.+7℥ 1*gr*. ?
4. What is the total weight of *A* 19℔ 5*oz*. 7*dr*.+15*cw*. 1*qr*. 15*oz*.+*T* 3 98℔ 6*dr*.+17*cw*. 34℔ 7*oz*. 3*dr*. ?
5. What is the total measure of *G* 5*qr*. 7*bu*. 1*ga*. 1*pt*.+3*bu*. 3*pc*. 1*qt*.+11*qr*. 2*pc*. 1*pt*.+6*bu*. 1*ga*. 3*qt*.+8*qr*. 3*qt*. ?
6. What is the total measure of *L* 1*p*. 1*bl*. 7½*ga*. 2*qt*.+1*t*. 3*hhd*. 20¼*ga*. 2*gi*.+3*p*. 61*ga*. 1*pt*.+1*bl*. 2*ga*. 2*qt*. 1*pt*. 2*gi*. ?
7. What is the total length of *E* 5° 65*ml*. 3*fu*. 1*ft*.+2*lg*. 7*fu*. 15¼*yd*.+13*ml*. 7*r*. 2*ft*. 9*i*.+68·07*ml*. 4*fu*. 19*r*. 23½*yd*. 5*i*. ?
8. What is the total length of *El*.8 2*qr*. 3*n*.+6*yd*. 1*qr*. 1*n*.+*El*.3 4*qr*. 2*n*.+17*yd*. 2*qr*. 1*n*.+*El*.28+25*yd*. ?
9. What are the total dimensions of 85*ac*. 3*rood* 15¼⌉*y*.+649*ac*.+26*ac*. 17⌉*r*. 3⌉*f*. 127⌉+2*rood* 27¼⌉*y*. 3⌉*f*. 61⌉ ?
10. What are the total dimensions of 30*cy*. 15*cf*. 1500*ci*.+28*cy*. 6*cf*. 1686*ci*.+23*cf*. 1703*ci*.+12*cy*. 26*cf*. 966*ci*. ?
11. What length of time is the entire of 1873*yr*. 155*da*. 3″+46*yr*. 38*da*. 5′+16*wk*. 6*da*. 12*h*.+226*yr*. 38*wk*. ?
12. What is the entire measure of 90° 43′ 17·18″+180° 56″+63° 16′ 42″+3° 25′ 48·76″ ?

Subtraction.

Is it ever necessary to increase a lower denomination by more than a unit from the higher, in the subtraction of composites? — When the unit of a higher denomination contains units and parts of a lower, the composite of a lower, increased in subtraction by a unit brought from the right, may exceed a single unit of the higher, and consequently require more in order to subtraction.

Can you exemplify the case? — If the subtractor be 31*ga*., increased to 32 by a unit from the right, and there be a cipher in the minuend, a single barrel will not suffice for the subtraction, a barrel containing only 31½ gallons.

Does it happen in no other case? — It may happen whenever there is a fraction, properly or improperly, in the subtractor; for it may then be increased, in a similar manner, to an amount exceeding a unit of the next higher denomination.

What is the rule?

RULE OF COMPOSITE SUBTRACTION.

To subtract composites, arrange terms of the same denomination one under the other, and subtract from the right. When needful, take a unit or more from a higher denomination, reduce, and add to the lower; subtract, and carry the units taken to the subtractor of the same name.

Proof, by composite addition; but this is not an independent proof either of a correct arrangement or reduction, as the same error may run through both the subtraction and addition.

APPLICATION.

		1.						2.		
Q 17	2*qr.*	0℔	0*oz.*	3*dr.*	minuend.	28⌉*r.*	0⌉*y.*	6⌉*ft.*	84⌉
9	3	27	8	11	subtractor.	13	30	6	123
Q 7	2	0	7	8	difference.	13⌉*r.*	29½⌉*y.*	8⌉*ft.*	105⌉

In the 2d example, the square yards of the subtractor, being increased to 31, cannot be taken from a single rod, or 30¼⌉*yd.*; two therefore are taken, and 2 carried to the subtractor of the same name. The proof requires recitation only, not figuring.

Examples to be wrought and recited.

What remains from

1. £57 5*s.* 0½*d.*—£56 9*s.* 11¾*d.*? £137 16*s.* 7*d.*—£100 18*s.* 10½*d.*?
2. ℥ 38 16*dw.*—21℔ 3*oz.* 19*dw.* 4*gr.*? 11*oz.* 9*dw.* 20*gr.*—19*dw.* 23*gr.*?
3. 2℔ 7℥ 3*gr.*—1℔ 4℥ 15*gr.*? 1℔ 6℥ 19*gr.*—10℥ 1ʒ 2℈ 19½*gr.*?
4. *A* 24℔ 3*oz.*—21℔ 9*oz.* 7*dr.*? Q5 1*qr.* 7*oz.*—5*cw.* 3*qr.* 8*oz.*?
5. *G* 6*qr.* 7*bu.* 1*qt*—6*qr.* 6*pc.* 3*ga.*? 28*bu.*—3*qr.* 5 *bu.* 1*pc.* 3*ga.*?
6. *L* 3*p.* 36½*ga.*—5*hhd.* 1*bl.* 8*ga.* 3*gi.*? 1*t.* 1*hhd.* 9*ga.*—2*p.* 31½*ga.* 3*qt.*?
7. *E* 1° 3*lg.* 4½*yd.*—54*ml.* 31*r.* 11*ft.*? 47*ml.* 3*fu.* 7*i.*—28*ml.* 39*r.* 3¼*yd.*?
8. *El.*3 3*qr.* 1*n.*—1*yd.* 3*qr.* 3*n.*? *El.*13 1*qr.* 2*n.*—5*yd.* 1*qr.* 3*n.*?
9. 1 sect. of U. S. land—⅛ sect. 3*ac.*? 170*ac.* 37⌉*r.*—¼ sect. 3*rood.* 20¼⌉*y.*?
10. 1 ton of freight—1*cy.* 1425*ci.*? 8*cy.* 20*ci.*—5*cy.* 1700*ci.*?
11. 1 century—78*yr.* 166*da.* 3*h.*? 69 solar years—18 civil years?
12. 1 circle—193° 15′ 5″? 1 quadrant—73° 29′ 31″? 1°—59′ 59.00834″?

Multiplication and Division.

What is the use of the arithmetic of composites? — Composite arithmetic supersedes the necessity of reduction to the lowest denomination expressed.

If pence be multiplied, how should the product be noted? — Pence, increased by multiplication to 12 or more, I would divide by 12, carry the quotient to the product of shillings, and set down the remainder under its own denomination.

In a distribution of shillings, how may the division of any remainder be continued? — A remainder from shillings must be reduced to pence, and from pence to farthings, that the division may be carried to the farthest possible extent.

Can the price of commodities be found under the rules for composites? — Composites are used in finding the price of commodities, because commodities are figured by composite numbers; but the consideration of proportion is necessary in every case of valuation whatever.

Can you now state the rule of multiplication?

RULE OF COMPOSITE MULTIPLICATION.

To multiply composites, set the multiplier under the lowest denomination; by reduction upward, raise the product of a lower denomination, when in excess, to the next higher; carry the quotient thereto, and set down any remainder under its own denomination.

If the multiplier be a prime number not tabular, take submultiples of the nearest smaller multiple, and to their product add the product by the difference multiplied into the original multiplicand; or reduce the multiplicand to the lowest denomination expressed, multiply, and then raise the product.

Proof, by composite division.

What is the rule of division?

RULE OF COMPOSITE DIVISION.

To divide composites, proceed as in the division of integers. By reduction downward reduce any remainder of a higher denomination to the next lower; and any of the lower expressed in the dividend, divide, and so proceed to the lowest.

Proof, by composite multiplication; but, should a fraction occur in the factors either of multiplication or division, the proof may differ in denomination, though not in value, from the original terms; and no proof of composites will certainly detect mistakes of memory in the tables.

APPLICATION.

1. What is the number of hogsheads in 39*ga.* 3*qt.* 1*pt.* 3*gi.*×75?

39*ga.* 3*qt.* 1*pt.* 3*gi.*×3
12×6=72

7	38	2	1	0
				6
45	42	3	0	0
1	56	3	1	1
47*hhd.*	36*ga.*	2*qt.*	1*pt.*	1*gi.* ans.

75 : 47*hhd.* 36*ga.* 2*qt.* 1*pt.* 1*gi.*= [39*ga.* 3*qt.* [1*pt.* 3*gi.* [proof.
9*ga.*×7=1*hhd.*

423
7
:2997
225
747
675
72
4*qt.*=1*ga.*
:290
225
65
2*pt.*=1*qt.*
:131
75
56
4*gi.*=1*pt.*
:225
225

The product of gallons is often too large to be obtained or reduced mentally; recourse must then be had to side work on the slate. In the first product above, the gallons are 479; to be divided by 63, that the denomination may be raised to hogsheads. The third product is that obtained by multiplying 3 into the original multiplicand; thus making up the deficiency left by 72 from 75.

In the proof, because the highest denomination does not contain the divisor once, we begin with reduction; and as a quotient arises it is placed, duly marked, on the right of the equation. The divisor is supposed to be repeated at every reduction, and the composites of the dividend are added to the first partial product or products obtained in reducing. It is however of importance to be recollected, that when submultiples are taken, as of 63*ga.* above, *the composites of the dividend must be added to the product of the last submultiple only;* for if added before, they also will be multiplied; whereas they are already in the denomination sought by reduction. Submultiples can seldom be used to advantage in the division of *composites;* in the present case they are altogether inapplicable.

2. How much are $\frac{3}{13}$ of Q 2 76℔ 15oz.?

Q 2 76℔ 15oz.
$\frac{3}{13}$

13 : 8 6 13=69℔ 7oz. $2\frac{6}{13}$dr. [ans.
112℔=1 Q

: 902
78

122
117

5
16oz.=1℔

: 93
91

2
16dr.=1oz.

: 32
26

6

69℔ 7oz. $2\frac{6}{13}$dr.÷$\frac{3}{13}$
13

3 : Q 8 6 13 0

Q 2 76℔ 15oz. proof.

The proof in this case is easily obtained by tabular division and mental reduction.

Examples to be wrought, proved, and recited.

What are the products of

1. £64 6s. 10d.×6? £175 11¼d×11? 19s. 10¾d.×108?
2. 5℔ 4℥ 1℈×7? 2℔ 9℥ 7ʒ×9? 3℥ 2℈ 19gr.×12?
3. *A* 23℔ 13oz. 5dr.×15? 17cw. 3qr. 15dr.×22?
4. *L* 3t. 3hhd. 20½ga.×84? 5p. 1hhd. 3qt.×$\frac{9}{14}$?
5. *El.*39 4qr. 2n.×76? 63yd. 3qr. 1n.×$\frac{5}{16}$?
6. 11 freight tons 13cf. 1200ci.×67?
7. 18° 54″×13·03? 11° 55·59″×·072?

What are the quotients of

8. *T* 30℔ 7oz.÷7? 53℔ 5oz. 19dw.÷18?
9. *G* 706qr. 5bu. 3pc.÷11? 38qr. 1pc. 1pt.÷36?
10. *E* 90° 61ml. 2ft.÷8? 8lg. 25r. 3¼yd. 5i.÷27?
11. 587ac. 2roods 3⌉f.÷13? 1 section of land÷7?
12. 66yr. 83da. 5h.÷52? 1 century÷93·5?

Duodecimals.

Whence the word *duodecimal*? — Duodecimal is from a Latin word signifying *twelve*.

Then what are duodecimals? — Duodecimals are numbers decreasing in the ratio of 12 to 1, as decimals in that of 10 to 1.

Under what part of arithmetic do they range? — They are fitly arranged with composites; for they are operated with as whole numbers, distributable into parts having invariable denominators and peculiar names.

To what uses are they applied? — They are of extensive application in calculating the square dimensions of superficial work, plank, &c., and the cubic dimensions of solid work, timber, freightage, and so forth.

Computed according to what tables? — Computed by the tables of square and cubic measure.

Is there any limit to their decrease? — Like decimals, they have no necessary limit; but usage limits their peculiar table to 4ths, or parts having the 4th power of 12 for their denominator.

How are they denominated? — Duodecimals are denominated feet, primes, seconds, thirds, fourths, &c.

Whence are these denominations derived? — Prime denotes the first and largest subdivision; second, third, and fourth, the succeeding ones and their order, correspondent with the powers of 12 contained in their denominators.

What denomination has the unit? — The foot square is the unit of duodecimals, and their highest denomination; for beyond, the scale of increase is not 12; and by the foot cubic or square, its products or parts, all solid and superficial work is estimated.

What is the next denomination? — A prime is the 12th part of a foot square.

What instrument is an example of it? — A foot rule, one inch wide, is, in superficial dimension, a prime; a 12th part of this superficies is a second.

Can you recite the table?

12'''' fourths	= 1 third.	(Marked) '''
12 thirds	= 1 second, or inch square or cubic.	''
12 seconds	= 1 prime, or 12 inches square or cubic.	'
12 primes	= 1 foot square or cubic.	$\lnot$*f*. or *cf*.

N. B. For the purposes of this rule, the inch is always to be subdivided duodecimally.

What general names are appropriate to the dimensions of a room? — The dimensions of a room are length, breadth, and height.

How are these names applied? — Length denotes the longest wall of a room; breadth, the shortest; height is that of either wall; length and breadth appertain also to ceiling and floor.

How are dimensions squared and cubed? — Dimensions, like all other numbers, are squared and cubed by multiplying one into another.

When fractions having the same denominator are multiplied together, what will be the denominator of the product? — The denominator of the product of two similarly denominated fractions will be the second power of their common denominator.

Can you exemplify it? — If $\frac{1}{12}$ be multiplied into $\frac{1}{12}$, the denominator of the product will be 12 times 12 ($\frac{1}{12}\times\frac{1}{12}$); and if again multiplied into the same, its denominator will be the third power of 12 ($\frac{1}{12}\times\frac{1}{12}\times\frac{1}{12}$).

Multiply feet into feet, what is the product? — Feet multiplied into feet, produce square feet; for both in superficial and solid measure, each unit of any one of the factors represents a square of the dimensions denominated; and these are repeated as many times as there are units in another factor; the first product therefore consists of squares denominated from the integral factors.

How will solids arise? — The squares, when produced, are then, for the purpose of a different measure, assumed to be solids, of a third dimension equal to the root of the squares found; and these are, by multiplication, repeated as many times as there are units in the third factor.

Feet multiplied into inches produce what? — Feet multiplied into inches produce primes; for the feet may represent a line of square feet, and of these a 12th part is taken for every inch in the fractional factor; but a 12th part of the foot square is a prime; the product consequently consists of primes.

Have you an instance? — We have it in the foot rule, which is a foot square taken to the extent of one twelfth, and producing a prime or 12 square inches.

How are duodecimals figured? — Duodecimals are figured, as other composites, by integers.

In the case mentioned then, how would the product appear? — A prime would be figured by the digit 1, in the place of primes, the denominator, 12, being understood.

Should the product be in excess, what must be done? — If the composites produced equal or exceed a unit of the higher denomination, they must be raised to the higher by division.

Multiply inches into inches, what is the product? — Inches multiplied into inches produce seconds; for multiplied into feet they produce primes; but a foot is 12 times the magnitude of an inch; the product therefore of inches only is only a 12th part as large.

What is obtained by the use of the duodecimal notation? — The advantage of the duodecimal notation is in the multiplication of composites together without reduction.

Does any thing particularly contribute to the facility of the operation? — The pointers over the inferior denominations direct at sight to the powers of 12 in the denominator of a product; for that denominator is the power of 12 indicated by the sum of the pointers over both factors, and their product is denominated from that sum.

Can you exemplify this? — The product of an inch (1′) multiplied into an inch (1′) is one second (1″); and the sum of the pointers over the factors is two.

When there are no pointers, what is the denomination? — The absence of pointers indicates the denomination of feet.

Is the product of factors varied by a change of denomination? — Composite factors, reduced to lower denominations, give the same product, in value, with the higher denominations; deficiency in the value of their units being compensated by increase of number.

Does any other mode offer itself of applying duodecimal factors? — Duodecimals may sometimes be converted into aliquot parts of the foot.

In what cases is this impracticable? — When an intermediate denomination is wanting in all the factors, aliquot parts cannot be taken, since the least denominator will then be 144.

Can you now state the rule?

RULE FOR SQUARING AND CUBING DUODECIMALS.

To multiply duodecimally, arrange dimensions as factors, feet under feet, inches under inches, lines under lines, noted by pointers to the left, one for every power of 12 in their denominators. Begin on the right, with the lowest denomination of the multiplier, and denominate every particular product from the absence or number of pointers in both particular factors, or multipliers actually operated with; when in excess, divide by 12, carrying the quotient to the denomination next higher to that indicated by the pointers, and setting down any

remainder on the right, pointed according to its true denomination in square numbers. Proceed thus to the highest denomination of the multiplier in a separate product for each denomination; the entire product is the square dimensions, and the multiplicand of any third dimension.

When a multiplier can be chosen, wanting no intermediate denomination, multiply one entire dimension into the highest denomination of another; then, for the lower, take aliquot parts in the higher, and point accordingly. The separate products added together form the square dimensions, and the multiplicand of any third dimension.

Proof, of one mode by the other; or of either, by reduction of the factors to the lowest denomination expressed, and subsequent raising of their product.

It has been customary to direct the operator to begin multiplying with the highest denomination of the multiplier; for no reason, one might imagine, except to increase the perplexing varieties of a most perplexing rule. The rule itself is usually stated in the form of maxims expressive of results; but a discernment of its nature, and the recollection, that products take their denomination from the absence or number of pointers in both factors, supersede the use of them. The maxims are also so expressed as constantly to keep up the erroneous idea, that inches and primes, seconds and lines, are identical, than which nothing can be more different; a prime being neither linear inch nor square inch, but twelve times as great as the latter; and a second being the same with a square inch.

APPLICATION.

1. What is the extent of painting in a room 20*ft.* long, 14*ft.* 6*i.* wide, and 10*ft.* 4*i.* high, deducting a fireplace of 4*ft.* 4*i.* by 4*ft.*, and 2 windows, each 6*ft.* by 3*ft.* 2*i.*?

By duodecimals.

3*ft.* 2′ width	4*ft.* 4′	14*ft.* 6′ width
6 height	4	10 4′ height
19 0′	17 4′ firep.	4 10′
2	38 0 windows	145 0
38 ⌉*f.* wind.	55 ⌉*f.* 4′	149 10′ short wall
		206 8 long wall
		356 6′
	10*ft.* 4′ height	2
	20 length	
		713 0′ the 4 walls
	206 ⌉*f.* 8′ long wall	55 4′ deduct firep. & wind.
		657 ⌉*f.* 8′ answer.

By reduction.

6*ft.*=72′ height

3*ft.* 2′=38′ width.
6′ × 12=72′

228
12

2736″
2

5472⌉ windows

4*ft.*=48′

4*ft.* 4′=52′
4′ × 12=48′

208
12

2496″ firep.
5472″ windows

7968⌉

20*ft.*=240′ length

10*ft.* 4′=124′ height
20′ × 12=240′

2480
12

29760″ long wall
21576″ short wall

51336
2

102672″ the 4 walls
7968″ deduct firep. & wind.

1⌉*f.*=144⌉ 12:94704″

12: 7892′

657⌉*f.* 8′ proof.

10*ft.* 4′=124′ height

14*ft.* 6′=174′ width
124′

696
2088

21576″ short [wall

A great deal of side work is commonly necessary in duodecimal operations; for example, in multiplying the width of the short wall by 4′, we cannot set down the first partial product, till we have got the second and third; for the first is 2 primes, the products to come are primes also, and these must be added together, before we can tell what finally will range under that denomination.

In solving this question, we do not take the ceiling into account, the work not being of a nature usually applied to ceilings; but if the question had been, How many square feet of plastering? the ceiling also must have been included.

2. What is the superficial extent of a piece of ivory, 1 inch 11 lines broad, 11 inches 6 lines long?

By duodecimals.

$11'\ 6''$	length
$1'\ 11''$	breadth
$10''\ 6'''\ 6''''$	
$11''\ 6'''\ 0''''$	
$1'\ 10''\ 0'''\ 6''''$	answer.

By aliquot parts.

$6''=\frac{1}{2}'$	$11'\ 6''$	
$3''=\frac{1}{4}'$	$1'$	
$2''=\frac{1}{6}'$		
	$11''\ 6'''$	
	$5''\ 9'''$	
	$2''\ 10'''\ 6''''$	
	$1''\ 11'''$	
	$1'\ 10''\ 0'''\ 6''''$	proof.

In the present case, the operation by aliquot parts does not recommend itself; it serves however as a mode of proof.

3. What are the solid contents of a wall, 20*ft.* in height, 17*ft.* 3*i.* in length, 18*i.* in thickness, exclusive of a doorway, 7*ft.* 2*i.* by 3*ft.* 4*i.*?

By aliquot parts.

$4'=\frac{1}{3}ft.$	7*ft.*	$2'$	height
	3		
	21	$6'$	
	2	$4'\ 8''$	
$6'=\frac{1}{2}$	23	$10'\ 8''$	
	11	$11'\ 4''$	thickness
	35*cf.*	$10'\ 0''$	doorway

$6'=\frac{1}{2}ft.$	17*ft.*	$3'$	length
	8	$7'\ 6''$	
	25	$10'\ 6''$	
	20		height
	517*cf.*	$6'\ 0''$	wall
	35	$10'\ 0''$	deduct doorw.
	481*cf.*	$8'\ 0''$	answer.

By duodecimals.

7*ft.*	$2'$	height
3	$4'$	width
2	$4'\ 8''$	
21	$6'\ 0''$	
23	$10'\ 8''$	
1	$6'$	thickness
11	$11'\ 4''$	
23	$10'\ 8''$	
35*cf.*	$10'\ 0''$	doorway

17*ft.*	$3'$	length
20		height
345	$0'$	
1	$6'$	thickness
172	$6'$	
345	$0'$	
517*cf.*	$6'$	wall
35	$10'$	deduct doorw.
481*cf.*	$8'$	proof.

In this example, the mode by aliquot parts appears evidently the best. Side work in the duodecimals of the thickness is avoided by mentally taking $\frac{1}{2}$ foot.

Examples to be wrought, proved, and recited.

1. What is the extent of a ceiling, 17*ft.* by 13*ft.* 4*i.*?

2. What is the superficies of a floor, 19*ft.* 5*i.* by 17*ft.* 3*i.*?

3. What are the square dimensions of a room, 12*ft.* 5*i.* high, 21*ft.* long, 18*ft.* 2*i.* wide?

4. How many square feet of painting are there in a room, 18*ft.* 6*i.* long, 16*ft.* 9*i.* wide, and 11*ft.* high; having 2 windows, each 5*ft.* by 3½*ft.*, and a fire place, 5½*ft.* by 4¼*ft.*?

5. What are the solid contents of a wall, 50*ft.* long, 11*ft.* 8*i.* high, 2⅓*ft.* thick?

6. What are the square dimensions of a plank, 37*ft.* 7*i.* 5' in length, 1*ft.* 9*i.* 2' in breadth?

7. What are the contents of a piece of hewn timber, 31¾*ft.* long, 1*ft.* 7' broad, 2*ft.* 3' thick?

8. What is the extent of plastering in a room, 13*ft.* 3*i.* wide, 14*ft.* 5*i.* long, 9½*ft.* high?

9. How many square yards of paving are there in 191*ft.* of length, 7⅓*ft.* of breadth?

10. What are the cubic dimensions of a bale, 4*ft.* 11¾*i.* long, 3⅛*ft.* wide, 4⅛*ft.* thick?

11. How many feet of wood in a load, 5¼*ft.* high?

12. What are the cubic contents of a reservoir, of the 3 dimensions following: 12*ft.* 7¼*i.*, 8*ft.* 11½*i.*, 7⅜*ft.*

13. What are the solid contents of 3 bales, severally of the dimensions following: No. 1; 3*ft.* 7*i.*, 2*ft.* 5*i.*, 1*ft.* 9*i.* No. 2; 6*ft.* 1*i.*, 5¾*ft.*, 4*ft.* 8*i.* No. 3; 8*ft.* 5*i.*, 3⅙*ft.*, 2⅚*ft.*?

14. What is the freight tonnage in the following cases: No. 1; 11*ft.* 5*i.*, 3*ft.* 4*i.*, 2*ft.* No. 2; 7*ft.* 3*i.*, 5½*ft.*, 1¼*ft.* No. 3, 15*ft.* 5*i.*, 4⅜*ft.*, 3⅔*ft.* No. 4; 16*ft.* 10*i.*, 9*ft.* 7*i.*, 3*ft.* 2*i.* No. 5; 10*ft.* 1*i.*, ⅞*ft.*, 2¾*ft.* No. 6; 14*ft.*, 7*ft.* 7*i.*, 5*ft.* 9*i.*?

15. How many yards of carpet, 19 inches wide, will cover a floor, 18*ft.* long, 13*ft.* 5*i.* wide? See ex. 2 (wrought), in Valuation.

The fractions of feet, &c., in these examples must be reduced to composites, when needful, and be so stated in the terms of the operation; should any of them be incapable of such a reduction, it must be ascribed to inadvertency, and the nearest composite be taken.

It is customary to bring the whole subject of duodecimals and their application under one head; we shall therefore subjoin examples in valuation, which must be passed over for the present, that the learner may first make himself acquainted with the master principle of mathematics, proportion.

In order to valuation, duodecimal parts of a foot in a result should be raised to decimals.

VALUATION.

1. What is the freightage of a case, $19ft.\ 5i.$ long, $3\frac{7}{8}ft.$ wide, $2ft.\ 11i.$ deep; at $26 per ton?

By aliquot parts.

$\left\{\begin{array}{l}\frac{4}{8}ft.=\frac{1}{2}ft.\\ \frac{2}{8}\ =\frac{1}{4}\\ \frac{1}{8}\ =\frac{1}{8}\end{array}\right.$	$19ft.$	$5'$ length				
	3	width				
	58	$3'$				
	9	$8'$	$6''$			
	4	$10'$	$3''$			
	2	$5'$	$1''$	$6'''$		
$\left\{\begin{array}{l}6'=\frac{1}{2}ft.\\ 4'=\frac{1}{3}\\ 1'=\frac{1}{12}\end{array}\right.$	$75 \rceil f.$	$2'$	$10''$	$6'''$		
	2					
	150	$5'$	$9''$			
	37	$7'$	$5''$	$3'''$		
	25	$0'$	$11''$	$6'''$		
	6	$3'$	$2''$	$10'''$	$6''''$	
Solid dimens.	$219cf.$	$5'$	$4''$	$7'''$	$6''''$	

$$5'\ 4''=\frac{5}{12}+\frac{4}{12\times12}=\frac{64}{144}=\frac{8}{18}=\dot{\cdot}4cf.$$

In a result, the fractions smaller than cubic inches are presumed to be of no importance.

The freightage sought is the ratio,

$\frac{219\cdot\dot{4}}{40}$ of $26, f. given; $=4,0 : 21\cdot9\dot{4}=5\cdot486$

$\left\{\begin{array}{l}\frac{26}{40}=\cdot65\\ \frac{142\cdot636}{219\cdot4}=\cdot65,\ \text{proof.}\end{array}\right.$

$$\begin{array}{r}5\cdot486\\ 26\\ \hline 32916\\ 10972\ \ \\ \hline \$142\cdot636,\ \text{answer.}\end{array}$$

Proof of the dimensions by duodecimals is left for the student.

2. What will be the cost of papering a wall, $12ft.$ long, $9ft.$ broad, with paper of $2\frac{1}{2}ft.$ wide, at $1·75 per yard?

$$\begin{array}{l}2ft.\ 6'\\ 3\\ \hline 7 \rceil f.\ 6'\ \text{dimensions of every}\\ \quad 1yd.\ \text{linear of paper.}\end{array}$$

$a\ r$ = $108 \rceil f.$ dimen. of wall.
$a\ e$ = $7\cdot5 \rceil f.$ dimen. of paper.
$e\ g$ = $1yd.$ linear of paper.

The length of paper sought is the ratio,
$\frac{108}{7\cdot5}$ of 1*yd.*; $=7\cdot5:108=14$*yd.* $1\frac{1}{5}$*ft.*

75	
330	
300	$\frac{1}{7\cdot5}$ ratio of length given.
30	$\frac{14\cdot4}{108}=\frac{1}{7\cdot5}$ ratio of l. sought. Proof.
3*ft.*=1*yd.*	
90	
75	
15	

The price sought is 14*yd.* $1\frac{1}{5}$*ft* × \$1·75=

	1*ft.*=$\frac{1}{3}$*yd.*	\$1·75
		7×2=14
$\frac{1\cdot75}{1}$ ratio of price given.		12·25
Ratio of price sought.		2
$\frac{35\cdot2}{14\cdot4}=1\cdot75$, proof.		24·50
	$\frac{1}{5}$*ft.*	·58$\dot{3}$
		·11$\dot{6}$
		\$ 25·200, answer.

In obtaining proof of the length, the decimal ·4 has been taken for the equivalent remainder, $\frac{2}{7\cdot5}$. The numerator of the same ratio of length is a measure of the denominator.

Examples to be wrought, proved, and recited.

1. What is the cost of paving 119*ft.* 2*i.*, by $7\frac{1}{2}$ *ft.*, at \$$12\frac{1}{2}$ per yard?

2. What is the expense of plastering a room, 23*ft.* 7*i.* long, 20*ft.* 9*i.* wide, and 12*ft.* 5*i.* high, at \$·25 per yard?

3. What is the price of a marble block, length, 5*ft.* 3*i.* 8', breadth, 4*ft.* 11*i.*, thickness, 4*ft.*, at a dollar per square foot on the superficies of every 6*i.* in thickness?

4. What is the cost of glazing the upper light of a green-house, 112*ft.* in length, 13*ft.* 5*i.* the slope, at \$·31 the foot square?

5. What is the cost of roofing a house, 40*ft.* in length within the walls, 28*ft.* 4*i.* breadth within, and the rafters $\frac{3}{4}$ of the breadth, at \$2 per square foot?

6. What is the freightage of a case, 11*ft.* 11*i.* long, 2*ft.* 7¾*i.* wide, 3½*ft.* deep, at $17 per ton?

7. What is the expense of levelling a bank, 6*ft.* 7*i.* in height, 5*ft.* 4*i.* in breadth, and 31*ft.* in length, at $·6¼ per cubic foot?

8. What is the cost of digging a reservoir, 8*ft.* deep, 7*ft.* 11*i.* square, at $·08⅚ per cubic foot?

9. What will be the cost of papering a room, 15*ft.* long, 13¾*ft.* wide, 7*ft.* 5*i.* high, with paper ¾*yd.* wide, at $·53 per yard?

10. What will be the cost of carpeting a room, 20*ft.* by 19*ft.*, with carpet 18*i.* wide, at $2 per yard?

11. What is the value of a load of wood, of legal breadth and length, 5¼*ft.* high, at $·93 per cord foot?

12. What is the value of a pile of cord wood, 32*ft.* long, 16*ft.* in height and breadth, at $·62½ per cord foot?

In addition to these, the preceding examples of dimensions only may be rated, and their entire values found. The learner will now proceed to inverse proportion, if the rules intervening have been studied.

Having now given what the author trusts is a correct and complete view of this most awkward of arithmetical operations, he presumes to express a wish, that the notation and its rule might be abandoned for ever. This however cannot be done till the foot shall be divided into tenths instead of twelfths; but, a beginning having already been made, by the subdivision of the *inch* into tenths, it may be hoped that the innovation will be extended to the foot, and the squaring and cubing of dimensions be reduced to the simple and more perfect process of decimal multiplication. In accomplishing this object, no master mechanic need wait for the consent of his brethren, one would suppose; let him order rules for himself, dividing the foot into tenths and hundredths, and the hundredths into halves (five parts or points in a thousand), and he will have no farther use for duodecimals. On the nature of duodecimals, the rationale of the rules for operating with them, the essential distinction that exists between the significancy of their denominations, and the same, as used in linear measure, no book which the author has looked into, and of which he has the recollection, affords the slightest information. To quote the words of a student to whom he is indebted for a valuable suggestion on the denominations of this rule, "he never before knew the point to be made a subject of discussion."

ARITHMETIC.

PART IV.

EVOLUTION.

What subjects are treated of in the Fourth part of arithmetic? — The Fourth part of arithmetic may comprehend involution, and its opposite, evolution.

What is the first mentioned? — Involution is the multiplying of a number into itself, or the raising of powers.

What is its opposite? — Evolution may be considered the disentangling of powers, and is the same with the extraction of roots.

What are roots? — Arithmetical roots are numbers producing, by involution, required powers; as 3, once multiplied into itself, produces 9, its 2d power.

Have all numbers roots? — Many numbers have no roots, not being themselves powers; because factors cannot be found, nor do exist, which, by involution, will produce them; but admitting of approximation within any important degree of accuracy, all numbers are said to have roots, perfect or approximate.

How is root distinguished from power? — Root and power are distinguished by their use and value only in any particular case; for the same number may be the root of one power, and be also the power of some other root.

Have you an example? — Twenty-five is the square root of 625, and is also the 2d power of 5.

Is there any example of root and power identified in the same number? — Unity is commonly described as both root and power; but this is an unintelligible statement, founded only in the manner of notation, it being impossible that unity, by involution, should produce any thing more or less than itself?

Yet are not powers of unity distinguished from one another? — The powers of unity, so called, are distinguished as composites, or by exponents, which are small figures raised above the line (1^2); these powers are assumed, or inferred, from the

nature of a question and the steps of an operation; being obtainable, in reality, from no root whatever.

Can you exemplify the case of inference? — When dimensions represented by unity are multiplied into each other, we infer the power from the number of involutions.

Will no fractional root produce unity? — No fraction can ever, by involution, produce a unit; for the oftener it is involved, the less it becomes.

What then do roots that are approximate consist of? — All approximate roots of integral or mixed numbers are themselves mixed numbers.

Have not fractions roots? — The roots of fractions are other and greater fractions, perfect or approximate, as the numbers specified are or are not powers.

Have you an example? — The root of ·25 is ·5; but of 5 itself, fraction or integer, a perfect root does not exist.

Can the smallest possible fraction have a root? — Numerically, there is no such thing as the smallest possible fraction; we may enlarge denominators, and diminish numerators, as much as we please; and either they will represent nothing, or possibilities only to an infinite power.

Can we take advantage of those possibilities? — We are gifted with faculties that enable us often to multiply those possibilities into actual results.

Of the Square Root.

What is the first root of a number called? — The first root of any number is called its square root; because equal length and breadth, which may form a square, are represented by equal numbers.

Of how many digits will the square root of a number having two places consist? — The root of any two figures forming a power must be a single digit; for it cannot be less; nor can it be more; for tens, or two places, produce, by one involution, 3 places at the fewest.

Of any number of digits, what will be the places in the root? — Of numbers paired from units, and forming a power, the square root will consist of one digit for every pair, and of an additional one for every single figure over; for a single figure may be a power, as 4 of 2.

Suppose the number not to be an integral power? — If the number specified be a mixed or pure decimal, the root will have one decimal place for every pair of decimals in the power, reckoning from either hand.

How does this appear? — Because the power is the product of equal factors, and has therefore twice the number of decimal places in either one.

Does this afford any indication of powers or the contrary? — *A mixed or pure decimal consisting of uneven places cannot be a square.*

Not if continued by ciphers? — Not if continued by ciphers; for ciphers on the extreme right of decimals are of no value, and no significant, multiplied into itself, can produce a cipher.

Suppose the number not to be a power? — If a number be not a power, its root will be approximate, and the true power of the root a mixed or pure decimal; which will therefore have two decimal places for one in the root.

How is an approximate root extended? — An approximate root is extended by pairing with ciphers the number of which the root is sought.

But if significants once involved can never produce a cipher, what purpose can this answer? — Ciphers are annexed to obtain the root of a number very nearly approaching to the number given; the number itself can never be produced from any root whatever.

In determining whether a mixed number be a power, from what place will you reckon? — The possibility of a mixed number being a power cannot be determined from the place of units; for, to be a square, the decimal places must admit of being paired among themselves; to be any other power, of being separable into corresponding portions.

What peculiarity attends the extraction of the roots of common fractions? — Common fractions may be reduced to decimals, and their roots may be evolved decimally; but if their terms be each a power, the roots of the terms are best evolved separately.

How are powers raised? — Powers are raised by involution; that is, by multiplication.

How are roots extracted? — Roots are evolved by division; for this is the opposite of multiplication.

In division, where do we begin? — In division we begin on the left.

What is the square of 13? — Thirteen 13s are 169.

Can you show, by reasoning, of how many figures the square root of 169 consists? — The square root of 169 has 2 places; for the places of the power are a pair, and a single place over.

How will you distinguish the pairs? — To pair the numbers, *a point* may be set over the right hand digit of each, and over

any single integer on the left, to distinguish it as giving a place to the root. $\dot{1}6\dot{9}$.

Calling the portions thus separated, periods, since they are not all pairs, what must be the square root of the first? — The square root of the first period in 169, is 1; for 1 only, in the ten's place, could produce it.

To raise the 2d power of 13 we shall now separate those digits into their component parts, of tens and units; the powers produced we shall exhibit by exponents, and other products by the annexation of factors; what appears to be the 2d term in the sum total of the 2d power of 13? — The 2d term in the 2d power of 13, consists of the left hand place of the root, twice taken and multiplied into the right hand place of the same.

$$\begin{array}{r}
10+3 \\
10+3 \\
\hline
\overline{10\times3}+3^2 \\
10^2+\overline{10\times3} \quad \\
\hline
10^2+\overline{2\times10\times3}+3^2 \\
\hline\hline
\end{array}$$

$$\begin{array}{l}
\sqrt{\dot{1}6\dot{9}}=13,\ \text{sq. root.} \\
\quad 1 \\
\hline
23:69 \\
\quad\ \ 69 \\
\hline\hline
\end{array}$$

To evolve then the 2d place of the root, what must we do? — To evolve the 2d place of a root, the first must be doubled, and made a divisor.

What further must be done? — Since the 3d term of the sum total contains a square number, the root of that number must be found, annexed to the left hand place of the divisor, and also to the quotient.

What will this root be? — This place of the root will be the number of times that the double of the first is contained in the new dividend; for it is a factor of it; it must be annexed to the divisor, because it is also a factor of the 3d term of the power, by multiplication incorporated with the 2d term; and to the quotient, because it results from the division, and is the 2d place in the root.

Can you now recite the rule?

RULE FOR THE EXTRACTION OF THE SQUARE ROOT.

To evolve the square root of a number, prefix the sign of the roots; then distinguish the digits into periods, by setting a point over every other, beginning at the place of units for integers, and at hundredths for decimals, paired completely by ciphers or repetends, if not otherwise pairs. Find, by trial, a square number equal to, or next less than, the first period, and set its root in the quotient; subtract as in division, and to any

remainder annex the succeeding period. Double the quotient already obtained, make it a new divisor, and consider how often it is contained in the new dividend, less the right hand place; insert this number in the quotient, annex it also to the divisor, multiply and subtract as before; and in this manner, for every additional place in the root, double the entire former quotient for a divisor, and annex thereto the new quotient figure.

When a new dividend does not contain its divisor once, annex a second period thereto, and a cipher to quotient and divisor; then find the new quotient figure as before.

Approximate roots are continued by the annexation, to preceding remainders, of pairs of ciphers or repetends.

Common fractions may be reduced to decimals of not fewer than 14 places, if approximate, and their square root taken in one operation.

The quotient is doubled for every new divisor, by doubling the right hand figure of the divisor preceding.

Contraction. When a large number of decimal places are desired in the root, after taking the first five significant decimals in the usual manner, double the last quotient for a divisor, and by it divide the last remainder in the manner of contracted division of decimals, abscinding a place from the divisor on every fresh division, till its digits are exhausted.

Proof, by involution.

APPLICATION.

1. What is the square root of 152399025?

$\sqrt{1\dot{5}2\dot{3}9\dot{9}0\dot{2}5}=12345$, answer.

```
      1
     --
   22 : 52
        44
       ---
  243 : 839
        729
       ----
 2464 : 11090
         9856
        -----
24685 : 123425
        123425
        ======
```

Proof obvious.

2. What is the square root of $\frac{5}{6}$?

$$\tfrac{5}{6} = \cdot 8\dot{3}.$$

$$\sqrt{\cdot \dot{8}\dot{3}} = \cdot 91287, \text{ answer.}$$

```
         81
         ----
181   :  233
         181
         -----
1822  :  5233
         3644
         ------
18248 :  158933
         145984
         -------
182567 : 1294933
         1277969
         -------
           16964
           =====
```

$\cdot 91287 \times \cdot 91287 = \cdot 8333316369$, proof.

The exactness of the proof at so early a stage of the division is owing to the power concluding with a repetend.

3. What is the square root of ·00032754?

$\sqrt{\cdot 0\dot{0}0\dot{3}2\dot{7}5\dot{4}} = \cdot 0180980667$, answer.

```
      1
      ---
28 :  227
      224
      ------
3609 : 35400
       32481
       ------
36188 : 291900
        289484
        ------
3,6,1,96 : 2416
           2171
           ----
            245
            217
            ---
             28
             25
             --
              3
             ==
```

```
  ·0180980667
  7660890810
  -----------
   18098067
   14478453
     162882
      14478
        108
         11
          1
 ------------
·00032754090, proof.
 ============
```

The conclusion shows, with a former instance, the unavoidable imperfection attending the mode of carrying the tens, as applied

to division, and the necessity of making a considerable number of places secure, before we begin to contract; yet it is not impossible, that inequality might exist on the right, in the places omitted. That the quotient figures obtained are correct, is manifest from the proof, which being exact, we may imagine that the numbers taken by us at random are an exact square; or some compensatory deficiency or excess in the abbreviated process may have given the appearance, without the reality, of exactness. The example affords evidence also of the facilities with which a very large number of decimals may be taken in a root, and proved.

Examples to be wrought, proved, and recited.

1. What are the square roots of 760? 995? 2000? 11? 1833? 37542? 163?

2. What are the square roots of 37·005? 516·9? 7034·111? 100·001? 15·3918?

3. What are the square roots of 11·3169? ·7919? ·08716? ·00261? ·0000605329?

4. What are the square roots of $\frac{1}{3}$? $\frac{2}{7}$? $\frac{2}{3}$? $\frac{7}{9}$? $\frac{1}{13}$? $\frac{9}{11}$? $\frac{5}{12}$? $\frac{12}{21}$? $\frac{11}{13}$? $\frac{13}{17}$?

5. Can a thousand men be formed into a perfectly square body? if not, what is the excess over the largest square possible?

6. What is the length of one side of a section of U. S. land?

7. If as many as possible of 5000 men be formed into a square, what number will there be in a rank?

8. If a nursery contain 2025 trees in rows of equal number every way, how many trees are there in a row?

9. A square room is covered by 530 square feet of carpeting; what is the length of a side?

10. A square building will accommodate nearly 1500 persons on the floor; what is the largest number that can sit in a row, averaging all at a size?

Of the Cube Root.

The extraction of the cube root is of so difficult investigation, and of so perplexed an operation, by any arithmetical rules, descriptions merely, as they are, of algebraic processes, that it seems a waste of time and labor to treat of them, while the method by logarithms is open to whosoever chooses to make himself acquainted with their use, and has need of the process. Yet, for purposes not very nice, it is possible, by a few multiplications, to approximate with sufficient nearness to the cube root. We select the example given in Hutton's Mathematics, and neglecting rules which no mortal can recol-

lect, shall endeavour to find, by trial, the cube root of 21035·8. The ancients were compelled to follow this mode.

The cube root of any three digits cannot exceed a single digit; for if the number be integral, then we observe, that 10, the least number of two places, produces a cube of four places, or 1000; if it be decimal, a single involution of a single significant will produce three decimal places. Further; of one or two integral places alone, or additional to ternary periods, there will also be one integral place in the cube root, for there cannot be less; since a fraction will never, by involution, produce a unit. Pointing therefore the integral part of the given number from units, according to this principle, we discover that its 3d root consists of two integral places; and to facilitate our operation, we begin with the least possible root having that number; this is 10. We proceed in our trials with the succeeding digits and a cipher, till we have discovered two cubes between which the number specified falls; then changing our course, at first by larger, and finally by smaller, approaches, we at length discover a cube differing so little from the cube in question, that we assume the perfect root of the one to be the approximate root of the other.

$\sqrt[3]{21035{\cdot}8} = 27{\cdot}605$, or for [thereabout;

$10^3 = 1000$
$20^3 = 8000$
$30^3 = 27000$
$25^3 = 15625$
$27^3 = 19683$
$27{\cdot}5^3 = 20796{\cdot}875$
$27{\cdot}6^3 = 21024{\cdot}576$
$27{\cdot}605^3 = 21036{\cdot}004$

Dr. Hutton makes the root to be 27·60491056; and from its product by involution we may pronounce it all but perfect; that product, as you may readily find by contracted multiplication, is 21035·799995. Such nice calculations however chiefly concern the higher mathematics; and it is evident, that by making one more trial, with the smallest possible reduction, namely of ·0001, for the very small difference that still existed, we might have considered ours the true approximate.

ARITHMETIC.

PART V.

PROPORTION.

Definitions.

What is considered in the last part of arithmetic ? — The subject of the last part of arithmetic is proportion.

What is proportion ? — ***Proportion in quantity is the comparative magnitude of things, their measures, or their numbers, one to another.***

How is the comparative magnitude of numbers expressed ? — Numbers are compared in fractional or dividual forms, called ratios.

What is meant by dividual forms ? — By dividual forms I mean the arrangement of terms as in common division.

What is ratio ? — ***Ratio*** is a Latin word of various signifition, used, among other meanings, to denote proportion.

Numerically, what is it ? — ***Ratio is the proportion between any two numbers.***

This is the quality ; what is the thing ? — ***A ratio is a numerical form expressive of proportion between any two numbers.***

What are quotients ? — Quotients also are ratios ; for, if ratios be proper fractions, nothing essential is changed by changing their form ; if they be improper fractions, by division a new but similar ratio arises, of unity to the quotient.

Can you demonstrate this by example ? — If the ratio be $\frac{1}{5}$, the quotient implies a ratio of 1 to 5 ; for the proportion deduced ($\frac{5}{1}$), and its verbal expression, five times as great or small, signify, that for every unit of one number, there are five of another.

How are ratios read ? — Ratios are read, as the denominator to the numerator, as the parts of the numerator in the parts of the denominator, as the divisor to the dividend ; or concisely, by terms of proportion ; as one half, two thirds, so many times greater or smaller.

Should fractional terms consist of several factors, what would be the ratio? — Whatever factors, or other members, there may be in a term, since these are reducible to a single number, the ratio is single, though, for distinction's sake, such ratios are sometimes named conjoint.

What are equal ratios? — Equal ratios are equal proportions, figured either as equivalent fractions, or as equal quotients.

How are they found? — Equal ratios are found by a proportional operation, termed, concisely, a proportion.

What then does the term *proportion* designate? — Proportion designates proportional operation, by which a new number or quantity is found; but denotes, chiefly, quantities placed in such a state of contrast as to indicate, upon inspection only, or division, how much greater or less one is than the other.

What are the requisites to a proportion? — In every proportion there must be two things, or numbers, at the fewest; for there is no comparison in unity.

Has proportion any limit? — Proportion is without limit, because comparison is so.

How is the proportion of two numbers found? — The proportion of one number to another is the same with the part that one is of the other; this is expressed in ratios, or found by actual division.

Can you exemplify it? — The proportion of fifty cents to a dollar is expressed by figuring the sum; for the sum ($·50) exhibits the parts, as being 50 in 100, a ratio which, reduced to its least terms, is of 1 in 2, or 2 to 1 ($\frac{1}{2}$).

What general distinction of numerical proportions exists? — Proportion is either equal, as between equal numbers, and similar ratios; or unequal, as in other cases.

Between what sort of things does it exist? — Proportion exists between things of the same kind, as between one sum of money and another, one quantity of goods and another of the same description.

Between things of the same kind only? — Proportion may exist between any kind of things, in respect of some quality or circumstance common to each, as of number, magnitude, power, value; thus we are constantly comparing money with goods.

What is the use of proportion in arithmetic? — The use of proportion in arithmetic is for the determination of estimates.

The author writes for the general student, and must therefore use popular language in its popular meaning. To observe the distinction made in books between ratio and proportion, would compel him to explain ratio by ratio, and make what he wrote unintelligible.

Estimates.

What is an estimate? — An estimate is a statement or computation of value, in money; of amount, quantity, or extent, needed and sought, in other things.

How are estimates determined? — Estimates are determined by a comparison of something already estimated with the article requiring estimate; in respect of the relation of each to the estimate itself.

Can you exemplify this? — If the value of any weight of tea be sought, it may be determined by reference to the value of a single pound; if the extent of carpeting necessary to cover a floor of known dimensions be the subject of inquiry, a comparison of those dimensions with the square contents of a single yard in length of the material, may furnish the answer.

Then what things are requisite to such a determination? — To form an estimate, two things must be known beside the article requiring estimate; namely, an article of the same kind actually estimated, and the actual estimate itself; in all, three things.

These being premised, what is the conclusion drawn? — Whatever proportion the articles estimated and requiring estimate bear to each other, the same must be the proportion of their respective values; in the same proportion must the amount be of the thing sought, and related to each.

Why so? — Because the greater or smaller the quantity, the greater or smaller is the sum to be expected for it; the greater or smaller the occasion of a demand, the greater or smaller is the amount necessarily to be supplied; so that half the quantity will produce half the money; treble the number of cattle require thrice the quantity of food.

Are estimates always thus made? — Estimates are arbitrary, or experimental, in the first instance; it being necessary to fix the value of some one thing or quantity, from which other things or quantities, of the same kind, can be rated.

How will you distinguish them? — *All estimates of price and quantity, at a certain rate, are determinable only by proportion.*

What are we to understand by rate? — *Rate is the estimate on some determinate amount or quantity, to which every other amount or quantity in the same kind is to be proportioned.*

How are proportions formed? — Proportions are expressed by ratios.

Then what is designed by rules of proportion? — Rules of proportion are designed for the determination of estimates, by the discovery of a term, which, with others specified, shall form equal ratios.

Why equal? — Ratios sought to be formed are ratios of estimate, and estimates are in the same proportion to each other, with their articles; the ratios therefore thus found are equal, if correct, to the ratios of the articles specified.

Of what nature are these articles? — *Articles estimated, and requiring estimate, are of the same nature and name;* or they could be of no *proportional* estimate.

Can you exemplify this? — We cannot determine the value of books from bricks; for their materials are different, and the labor bestowed on them exceedingly unlike.

In what consists the distinction of nature and name in these questions? — The distinction of *name* is absolutely necessary in all questions in which articles and estimates are of the same nature, as is frequently the case in money transactions.

What are the ratios expressed? — The ratios expressed, in questions of proportion, are of the articles; for things of the same nature and name are most fitly compared together.

Not of article and estimate? — Of article and estimate also in respect of value; but as an estimate remains to be found, these are best compared in the proof.

Then to the discovery of a ratio what is necessary? — To form an equal ratio, in a proportional operation, one ratio is, necessarily, to be known; and to find a proportional estimate, an actual estimate must be given.

How many terms are there, and how named? — Articles estimated and requiring estimate, with the estimate given, are the three known terms of a proportional operation; or, as generally expressed, of a question in the rule of three.

What is the result of such an operation? — Proportional operations result in the discovery of a fourth term, which is the estimate sought, and, with the estimate given, forms a ratio equal to that of the articles, taken in the same order.

Of what nature is the 4th term? — *The estimate sought is of the same nature and name with the estimate given, and comes out in the same denomination.*

Why of the same nature and name? — This is implied or expressed in the question itself; for the articles are compared together in respect of an estimate common to both, in nature and name; differing only in amount or value.

Why of the same denomination? — The estimate sought

appears in the same denomination with the estimate given, because it is a product of it.

How is this shown? — The proportion of the estimate given, assignable to the article requiring estimate, is the same with the proportion which the same article bears to the article estimated; this proportion is expressed by the ratio of the articles; that ratio therefore, and the estimate given, are factors of the estimate sought.

Can you exemplify this? — If 4℔ of tea cost \$6, three pounds will cost $\frac{3}{4}$ of \$6; for the ratio, $\frac{3}{4}$, expresses the proportion of 4 to 3. $\frac{3}{4}$ of $4=\frac{12}{4}=3$.

How are equal ratios thus produced? — The ratio of the articles, together with the estimate given, are factors of the estimate sought; consequently, if the first estimate be made a divisor of the second, the quotient will be the ratio of the articles; but a quotient is a ratio; therefore the ratios of the articles to each other, and of the estimates to each other, are equal ratios.

Can you exemplify this also? — Three fourths of \$6 are \$4·50; and the ratio of the estimates, $\frac{4\cdot5}{6}=\cdot75=\frac{3}{4}$, the ratio of the articles.

Must the articles correspond in denomination? — *Articles estimated, and requiring estimate, must be compared in the same denomination.*

What renders this necessary? — If the articles estimated and requiring estimate be of different denominations in the operation, the difference of denomination is equivalent to a difference in absolute quantity; equality being thus rendered apparently unequal, and inequality greater or less than the reality; the estimate found will therefore exceed, or fall short of, a true estimate.

Can you exemplify this? — If 25℔ of sugar be purchased at so much per cwt., the denominator of the ratio must be operated with as 100℔, not as 1*cw.*; or a quarter of a hundred will give 25 times the value of a hundred entire; which is absurd.

Kinds of proportion.

Is proportion of more than one kind? — Proportion is either direct or inverse.

What is the former? — Direct proportion is a like increase or diminution of estimate and of the article to be estimated; called direct, because estimate seems to follow article in a direct course.

How is it spoken of? — Proportion is said to be direct, when *more requires* more, or less requires less.

What is the latter kind? — Proportion inverse is an increase of estimate with a diminution of article, a diminution of estimate with increase of article; called inverse, because the estimate proceeds contrariwise to its article.

In what class of cases does it arise? — It applies chiefly to cases where time is to be gained by the increase of labor.

How is it spoken of? — Proportion is said to be inverse, when more requires less, or less requires more.

Single proportion.

Are there other distinctions in proportion? — Proportion is single or conjóint.

In what do they differ? — Single proportion implies a single comparison, or the use of a single ratio only in a proportional operation; conjóint, the use of two or more ratios.

Can you exemplify these distinctions? — Goods at a certain rate are in single proportion; namely, to other like goods; interest, at so much per cent. per annum, is in conjóint proportion; namely, of principal money to principal, and of time to time; different ratios however may be united in one, and treated as one.

When the value of a single thing is required, how is it found? — *When the numerator of a ratio is unity, in the same denomination with the denominator, divide the estimate given, by the article estimated; the quotient is the estimate sought, in terms of the estimate given.*

Why should this be so? — Because the quotient is in proportion to a unit, and whatever part the unit is of the article, the same part of the value is assignable to it.

When the article estimated is a unit, what is the process? — *When the denominator of a ratio is unity, in the same denomination with the numerator, multiply the article requiring estimate, into the estimate given.*

What principle makes the operation thus concise? — Evidently, so many times an article must be valued at as many times the estimate of that article.

Are these less cases of proportion than others? — On the contrary, every other case carried out by division is reducible to this; the ratio formed by the article estimated and the estimate given being as unity to their quotient.

What other considerations are there in aid? — The idea, that so many things are worth as much more than one thing, is altogether an idea of proportion; we can hardly express ourselves on the subject without introducing the word.

Are there not still others bearing on the same or similar

cases? — It is only from a consideration of proportion, that we learn the necessity of reducing articles compared to the same denomination, though estimated by the unit; and infer that the estimate sought issues, on division of one article by the other, in a quotient of the same nature, name, and denomination, and subject to the same reductions to higher or lower values, with the estimate given.

What then is the process in the latter case? — *When the estimate given is a unit, divide the article requiring estimate, by the article estimated, in the same denomination; the quotient is the value or amount sought, in terms of the estimate given.*

Can any other equal proportion be found in these operations? — *Articles compared are in equal proportion to their respective estimates, given and found, constituting similar ratios.*

How is this demonstrated? — The sole purpose of every proportional operation is to find an estimate, that shall be in the same proportion to its article, as the estimate given is to the article estimated; consequently, if this prove otherwise, the work, or the statement, is erroneous.

Have you an instance? — In the late example of 3℔ proportioned to 4℔ for $6, the ratio formed of the article requiring estimate, with the estimate found, is equal to that formed by the article estimated and the estimate given; for $\frac{6}{4}=1{\cdot}5$, and $\frac{4{\cdot}5}{3}=1{\cdot}5$; that is, the estimate of the unit is the same in both comparisons.

How is this equality of ratios produced by the work? — In the operation, the denominator of the ratio is a divisor of the estimate given; in the operation also, the numerator of the ratio multiplies their quotient; consequently, when the same numerator is made divisor of the product, or estimate sought, the quotient reappears; and first and second quotients are equal, for they are the same.

Then which article forms the denominator of the ratio? — *The denominator of the known ratio, in a proportional operation direct, is the article estimated.*

How is this demonstrated? — The estimate given, divided by the article estimated, shows the value of a unit in the denomination of that article; for whatever part the unit be of the entire article, that same part of the value or amount is assignable to the unit.

What is the inference? — In a proportional operation therefore, the article requiring estimate, if multiplied into the value of the unit, will produce the required value; and the article

estimated being necessarily made a divisor, in order to show the value of the unit, must be made the denominator of any ratio formed by the articles for that purpose.

Why did you limit your maxim to proportion direct? — In proportion inverse our language implies that operation or statement must be inverted.

What is the fact? — *The denominator of the known ratio, in a proportional operation inverse, is the article requiring estimate.*

How is this shown? — In proportion direct, by making the article estimated the denominator, the estimate sought will be as much larger or smaller than the estimate given, as the denominator is larger or smaller than the numerator; by making the article requiring estimate the denominator, the greater that article, the less will be the estimate; and the smaller the article, the greater will be the estimate; and this is proportion inverse.

What equal ratio will this afford? — *The inverse ratio of the articles is equal to the ratio of the estimates direct thereto.*

Can you demonstrate this? — The articles are inverted in forming the ratio; but, when formed, its operation is the same with that of a ratio direct; its results therefore are the same: these are, a ratio of the estimates direct to itself, and equal to itself.

Can you exemplify it? — If a single faucet empty a cask in 36 minutes, 3 faucets, opened on the same level, will empty it in $\frac{1}{3}$ of $36'=12'$. $\frac{1}{3}=\frac{12}{36}$, proof.

What is the application? — Here, $\frac{1}{3}$ is the ratio inverse of the articles, for the denominator, 3, represents the article requiring estimate; and $\frac{12}{36}$, the ratio of the estimates, taken in an order contrary to that of their articles, is yet equal to $\frac{1}{3}$, and therefore direct to $\frac{1}{3}$.

How in a contrary order? — In the statement, the article requiring estimate is made the denominator of the ratio known; in the proof, its estimate is made the numerator of the ratio found; and so of the two other corresponding terms.

In what sense direct? — The ratio of the estimates is direct to the inverse ratio of the articles, because the greater term of the one holds the same place with the greater term of the other; and if it were not so, the ratios could not be equal.

Conjóint propórtion.

How is a single comparison made? — The comparison of two numbers only is made by a single ratio.

Should it be necessary, in adjusting a proportion, to compare more than two numbers, what will be the ratios? — There will be as many ratios as there are comparisons; for a ratio is the proportion of two numbers only; but different ratios may be conjoined in one, and operated with as one.

In what manner conjoined? — Different ratios may be united in one, as submultiples of the same number; for all are to be multiplied into the estimate given, the estimate sought being derived through all the articles compared.

Can you demonstrate this by example? — The case of interest is the most frequent in conjóint proportion; for the amount of interest depends on the principal sum lent, and the time for which it is lent.

What is the manner of derivation? — An estimate thus to be obtained, when proportioned to one article, must then be proportioned to another; or the ratios may be multiplied into each other, and their product into the estimate given.

Why may they not be added together? — Conjóint ratios must not be added, but multiplied together, because one proportion actually proportionates the other; so that the estimate which one would give, the other halves, thirds, trebles, quadruples, and so on.

Can you show this by example? — Treble the principal gives treble the interest; but half the time, by severing the principal, if multiplied into it, halves the interest.

What will the different ratios consist of? — Each ratio will be formed of articles of the same nature and name, one already estimated, the other to be estimated.

What equal ratios does conjóint proportion eventuate in? — Conjóint proportion direct affords the same equal ratios with single proportion; for the different ratios are but factors of the same product.

Why do you say direct? — Conjóint ratios may exhibit proportions both direct and inverse; but, in this case also, if we look at the operation only, the ratios are equal and direct; the inversion affecting only the manner of making up the statement.

What is a statement? — *A statement is a symbolic representation, by figures and other arithmetical signs, placed in a certain order according to rule, of the terms and conditions of an arithmetical question.*

Can you exemplify it? — The late example in proportion inverse may suffice: If a single faucet empty a cask in 36 minutes, in what time will 3 faucets, on the same level, empty it?

How do you apply this? — The statement is, $\frac{1}{3}$ of 36′: here, the ratio and the estimate given form the three terms of the question; and the word *of*, for the definition does not *exclude* words, forms the only condition; namely, that the proportion expressed by the ratio shall be taken out of the estimate.

Might not the word be symbolically represented? — The word *of*, in arithmetic, commonly denotes multiplication, and whenever it does, the symbol may be inserted in place of it; the statement will then be altogether symbolic, and be read, one third multiplied into, or taken out of, 36 minutes.

$$\tfrac{1}{3}\times 36' = 12'$$

What name may we give to conjóint proportion direct and inverse? — Proportional operations uniting both direct and inverse ratios may perhaps be called mixed proportion.

How do these various distinctions bear on the importance of rules of proportion? — The distinctions of inverse and conjóint proportion involve such niceties of reasoning, that without rules of proportion, beside the loss of time, our perplexities would often be inextricable.

Does any argument arise out of the nature of the subject? Exactness is of the essence of proportion; without rules, there can be no exactness; therefore, without rules, expressed in some cases, and implied in all, there can be no proportion.

The observation is peculiarly applicable to cases where a per centage is to be added to, or deducted from the hundred, as in discount proper, and profit and loss.

Definitions of the different kinds.

Can you define the different kinds of proportional operation? — *Single proportion is the mode of determining an estimate when a single comparison is made between articles of the same nature and name, requiring therefore but a single ratio.*

Conjóint? — *Conjóint proportion is the mode of determining an estimate, by the joining together of two or more ratios in one, when more than a single comparison is made.*

Inverse? — *Inverse proportion is the mode of determining an estimate which increases or diminishes contrariwise to the article requiring estimate.*

Can you now recite the rules?

RULES OF PROPORTION.

Single direct.

To proportionate directly, make the article estimated the denominator, the article requiring estimate the numerator, of a ratio; both in the same denomination, or to be reduced thereto; multiply this ratio into the estimate given; the product is the estimate sought, in terms of the estimate given.

When the article estimated is a unit in the same denomination with the article requiring estimate, multiply the estimate given, into the article requiring estimate; the product is the estimate sought.

When the article requiring estimate is a unit, in the same denomination with the article estimated, divide the estimate given by the article estimated; the quotient is the estimate sought.

The division, in this last case, is really fractional multiplication, and ought so to appear, by a statement and operation such as may be seen in our first exemplification of profit and loss.

Proof. The articles estimated, and requiring estimate, are in equal proportion to their respective estimates, given and found, constituting similar ratios.

Inverse.

To proportionate inversely, make the article requiring estimate the denominator of a ratio, and proceed as in proportion direct.

Proof. The inverse ratio of the articles is equal to the ratio of the estimates direct thereto.

Conjóint.

To conjoin direct proportions, make the different articles estimated so many factors in the denominator, the articles requiring estimate in like manner factors in the numerator, of a ratio, the contrasted articles of which are all in the same denomination; simplify, reduce to the least terms, and multiply into the estimate given.

Proof. The conjoint terms are in equal proportion to their respective estimates, given and found; constituting similar ratios.

To conjoin proportions direct and inverse, make inversely proportionate articles requiring estimate, factors in the conjoint denominator; and their articles estimated, factors in the conjoint numerator.

Proof. The mixed ratio of the articles, as it appears in the operation, is equal to the ratio direct thereto, of the estimates given and found.

Is it possible to enuntiate the terms of a proportion in some concise manner? — Articles estimated might be designated by the mention and writing of their initial letters, *a e*; articles requiring estimate, by the letters *a r*; the estimate given, by *e g*; and the estimate sought, by *e s*.

Would this enable us to abridge the rules of proportion? — By substituting these initials for the names themselves, the rules may be brought within the compass, each, of a line or two.

Can you recite any such?

PROPORTIONAL MAXIMS OF EASY RECOLLECTION.

Single direct.

The estimate sought in proportion direct is the ratio, $\frac{a\,r}{a\,e}$ (denominated alike), of the estimate given.

Inverse.

The estimate sought in proportion inverse is the ratio, $\frac{a\,e}{a\,r}$ (denominated alike), of the estimate given.

Conjóint.

The estimate sought in conjóint proportion is the ratio, $\frac{ar \times ar}{ae \times ae}$ (denominated alike), of the estimate given.

Mixed.

The estimate sought in mixed proportion is the ratio, $\frac{a\,r \times e}{a\,e \times r}$ (denominated alike), of the estimate given.

The latter maxims should be read without the symbol $\times$; and the ratios of all the maxims, not by syllables, but by single letters. The literal ratios are designed to form a symbolic representation, which, aided by the maxim, may for ever imprint the great principle of proportion on the mind; and if that of proportion direct be there, all the others will follow.

Can you recite the general maxims?

GENERAL MAXIMS IN PROPORTION.

1. Articles estimated and requiring estimate are of the same nature and name, and must be compared in the same denomination.

2. The estimate sought is of the same nature and name with the estimate given, and comes out in the same denomination.

3. In proportional language, the word *as* precedes, and the word *to* follows, the denominator of a ratio: this, being the divisor of the two other terms in a proportional statement, may divide either one of them, or their product, as most convenient.

APPLICATION.

1. Seven pounds of tea, at \$1·50 per ℔; what is the price of the whole? $\$1{\cdot}5 \times 7 = \$10{\cdot}50$, ans.

2. Nine pounds three ounces of sugar, at \$·06¼ per ℔; what is the price of the whole?

The price sought is the ratio, $\frac{9℔\ 3oz.}{1℔}$ of \$·06¼, rate.

$2oz. = \frac{1}{8}℔$ \$·0625
9℔
·5625
$1oz. = \frac{1}{2}$ of $\frac{1}{8}℔$ ·0078125
·00390625
\$·57421875, ans.

$\left\{ \begin{array}{l} \frac{\cdot 0625}{16oz.} = \cdot 0039 \\ \frac{\cdot 5742}{147oz.} = \cdot 0039, \text{ proof.} \end{array} \right.$

A dealer would probably get the answer, by deducting a fourth from the value of ¼℔; but examples are not the less instructive for being in trifling matters. Words such as are prefixed to the work it may be unnecessary to insert in every instance; yet, after reading the question, every recitation of the work should begin with a similar application of the appropriate maxim.

3. The property in a town is estimated at \$126000, on which a tax is to be levied of \$1364; what is the rate per cent.?

$a\,e = \$126000. \quad a\,r = \$100. \quad e\,g = \$1364.$

$\frac{100}{126000}$ of \$1364, rate; $= \frac{1}{1260} \times 1364 = 126{,}0 : 136{\cdot}4 = \$1{\cdot}0825$ [ans.

$\left\{ \begin{array}{l} \frac{1364}{126000} = \cdot 0010825 \\ \frac{1\cdot 0825}{100} = \cdot 0010825, \text{ proof.} \end{array} \right.$

126
1040
1008
320
252
680
630
50

A difficulty in distinguishing the articles and the estimate, is best overcome, by considering carefully the numbers as they present themselves in the question, and setting them down, with their initial letters, in some orderly manner. It will also contribute to distinctness, in every instance of figuring a ratio, to annex the proper appellation of *e g* to the work. *Let it be observed once for all, that ratios susceptible of an easy reduction to smaller terms, must always be so reduced.*

4. A wall that is to be built to the height of 27*ft.*, was raised 9*ft.* by 12 men in 6 days; how many are required to finish the wall in 4 days?

a r=27*ft.*—9*ft.*=18*ft.*
a r=4*da.* inverse.
a e=9*ft.*
a e=6*da.* inverse.
e g=12 men.

e s is the mixed ratio, $\frac{18\times6}{9\times4}$ of 12 men; $=\frac{18\times12}{6}=36$ men, ans.

$\left\{ \begin{array}{l} \frac{18\times6}{9\times4}=\frac{18}{6}=3 \\ \frac{36}{12}=3 \text{ proof.} \end{array} \right.$

Few statements could present greater difficulty to the young arithmetician than the present; the most attentive consideration of the conditions of the question is requisite to disentangle the articles, and to assign to them their proper station. In forming the ratio, we have to consider, that the greater the quantity of work to be done, the greater is the number of hands required; here then is a direct proportion; but when time is limited, the number of hands must be increased as the time is diminished; this is an inverse proportion; the two, in combination, form what we have called a mixed ratio. After this, the work is so plain as to need no comment.

5. If 11 workmen earn \$137½ in a fortnight, what will be the wages, at the same rate, of 39 workmen during 5½ days?

e s is the conjoint ratio, $\frac{39\times5\cdot5}{11\times14}$ of \$137·5, wages given=

$\frac{39\times\cdot5}{14}\times137\cdot5=$ 137·5
19·5
6875
12375
1375
2 : 2681·25
7 : 1340·625
\$191·5178, &c. ans.

$\left\{ \begin{array}{l} \frac{137\cdot5}{11\times14}=154 : 137\cdot5=\cdot892 \\ \frac{191\cdot5178}{39\times5\cdot5}=214\cdot5 : 191\cdot5178=\cdot892, \text{ [proof.} \end{array} \right.$

a e=11 workmen.
e g=\$137·5
a e=14 days.
a r=39 workmen.
a r=5·5 days.

Such a question as the present probably never occurs in real life; and, if it should, would be solved by analysis, as the name now is; there is no inversion in it, as the greater the number of days, the more *wages* are due.

Although the questions determinable by proportion divide themselves, for the most part, into particular branches, treated of separately in the following pages of our work, there are a few that would be excluded from all; specimens of such we shall now introduce under one general head. Questions in conjoint and inverse proportion will be found under those titles.

Single Proportion Direct.

Examples to be wrought, proved, and recited.

1. At $1¾ per week, what will a twelvemonth's board amount to?

2. At $2·25 per week, how long can I be boarded for $200?

3. If my income amount to $373, and I spend $·75 per day, what shall I have saved by the end of the year?

4. If I borrow $500, during 3 months, without interest, what length of time shall my friend retain $413·195, in compensation for his kindness?

5. If bread be at $·04½ the pound, when flour is sold at $7 the barrel, what should the price per pound be, when the barrel of flour is sold for $8·25?

6. If a journey may be accomplished in 3 weeks, computing the length of the day as at the summer solstice, what difference will there be in the completion of it, if made at the equinox, and proportioned to the length of the day at that period?

7. At $4·80 the pound sterling, what is the value in federal money of £367 17*s.* 5½*d.*?

8. What is the amount sterling equivalent to $977·096, at $4·44̇4 to the pound?

9. A salary of $800 gives what per diem?

10. How much may a person spend per day, out of a salary of $1000, to enable him, during the year, to pay a debt of $265·63?

11. If a staff, 5*ft.* long, cast a shade on level ground, of 7½*ft.*, what is the height of a tower which, at the same moment, and on the same level, casts a shade of 200*ft.*?

12. How long will ¼ ton of bread last a ship's company, at 1½℔ per day?

13. If the quantity of water on board a ship be 1200 gallons, what may be the allowance to a crew of 20 men, computing the length of the voyage to be 120 days?

14. How many weeks will 750℔ of beef last a crew of 11 men, at 3½℔ each per week?

15. What number of spoons may be made from $1\frac{3}{8}$lb of silver, each spoon weighing $1\frac{1}{2}$ ounce?

16. If a ship sail four knots per $\frac{1}{2}'$, the knot being computed at 50*ft.*, and the mile at 6120*ft.*, what distance in miles will she make in the 24 hours?

17. If 3 tons 763lb of cotton are to be packed in bales, containing each 400lb, how many bales will there be?

18. A quantity of raisins, weighing cwt. 7 1 *qr.* 23lb 6 *oz.*, is to be packed in boxes, containing $15\frac{1}{2}$lb each; how many boxes are required?

19. If 7 tons of hay be food sufficient for 3 oxen during the winter, what weight is required to feed 17 oxen?

20. If an empty cistern, of 500 gallons capacity, receive 60 gallons of water per hour, and there be a leakage of $7\frac{7}{8}$*ga.* per hour, in what time will it be filled?

21. A piece of silver plate cost \$276·75, rated at \$1·63 per ounce; what was the weight of it?

22. If a pasture will maintain 21 oxen during 6 months, how long will it feed 121 oxen?

23. If a person rated at \$3117, pay \$$26\frac{1}{4}$, what should he pay who is rated at \$317?

24. What must be the length of an acre of land that is staked out $15\frac{1}{2}$ rods in width?

Tare and Tret.

How are weights spoken of, in reference to package? — The terms of commerce respecting package, and allowances therefor, are gross weight, tare, suttle, tret, and net weight.

Are you acquainted with the origin of any of these? — *Gross*, *tare*, and *net*, are French words, used by the French in commerce, to signify the same things as with us.

What is understood by gross weight? — Gross weight is the entire weight, including packing materials.

What is tare? — Tare is an allowance for the presumed weight of packing materials, when this cannot actually, or conveniently, be ascertained: it is made by deduction from the gross weight, at so much the single package, or by the cwt., or on the entire weight in a single amount.

What is tret? — Tret is an allowance, nearly obsolete, of 4lb in every 104lb, for waste on emptying packages, and on resale.

What is the weight remaining? — Net weight is the weight of goods, clear of all deductions; but when tare and

tret both are allowed, the remainder, after deduction of the tare only, is called the suttle weight.

Tret is probably from the Latin *trita* (Johnson), waste; *suttle* looks very much like a pun on the word, *subtle;* that is, not gross.

How are tare and tret determined? — Tare and tret are governed by the rule of single proportion direct.

What is the process? — When tare is rated at so much the package, the tare, multiplied into the number of packages, produces, of course, the tare sought; a gross weight consisting of hundreds, multiplied in like manner, must produce the tare entire, when rated by the cwt.

When the gross weight does not consist of exact hundreds? — Gross weight in two or more denominations may be multiplied into the tare, by the cwts. as integral composites, and by aliquot parts of the cwt. for the lower denominations.

Is there no other method? — The English cwt. consisting of 112℔ will frequently afford but very inconvenient aliquot parts, sometimes it affords none; in this case, the lower denominations may be reduced to decimals of the cwt., or a statement may be formed according to the rule of proportion.

What is the arrangement of the factors in the composite method? — The tare is the true multiplicand in every case; and, in all cases of more denominations than one, must be made the actual multiplicand; for the product must consist in denominations of the tare, not of the gross weight.

Can you exemplify this? — If the tare be ℔s, and the gross weight cwts., the product must be in ℔s, and parts of a ℔.

How is tare regulated in matters of public concern? — Custom-house tare is, for the most part, previously fixed, on averages calculated not to injure the merchant; and is expressed in tables, open to public inspection.

How may tret be found? — Since tret is an allowance of 4℔ on every 104℔ of the suttle, the proportion to be deducted from the entire suttle weight is $\frac{4}{104}=\frac{1}{26}$; multiplying therefore the suttle by $\frac{1}{26}$, and subtracting the product, we obtain the net.

Can you condense this subject into maxims?

MAXIMS IN TARE AND TRET.

Tare sought at so much the package, or cwt., is the product of the given rate multiplied into the number of packages, or clear hundreds; and into aliquot, or decimal, parts of the cwt., for lower denominations.

Proof, by equal ratios, as in proportion direct; or of *composite by* decimal multiplication.

Tret sought is one twenty-sixth part ($\frac{1}{26}$) of the suttle weight; to be deducted therefrom.

Suttle weight is the gross, less the tare.

Net weight is the gross, less every allowance actually made.

APPLICATION.

1. What is the net weight of a barrel of sugar, weighing cwt 2 3 *qr.* 13℔ gross, supposing tare and tret to be allowed, tare at 12½℔ per cwt.?

By composites.		*By decimals.*
2*qr.*=½ cwt	12·5 tare per cwt.	112 : 13·0=·11607℔
	2 cwt	cwt 2·75
	25·0	2·86607
1*qr.*=½ of ½	6·25	12·5 tare per [cwt.
7℔=¼ of ¼	3·125	1433035
4℔=$\frac{1}{7}$ of ¼	·78125	3439284
2℔=½ of $\frac{1}{28}$	·446428	35·825875℔ proof [of tare.
	·223214	
	35·825892℔ tare.	

cwt 2 3*qr.* 13℔	gross		
1 7·82589	deduct tare		10*qr.* 5·17411℔
2 2 5·17411	suttle		28℔=1*qr.*
10·96823	deduct tret		$\frac{1}{26}\times 285{\cdot}17411=10{\cdot}96823$℔
cwt 2 1 22·20588	net. Ans.		

2. Seven bags of wool; entire gross weight, 1676℔, whole tare, 36½℔; what was the tare per cwt. American?

Tare sought is the ratio, $\frac{100}{1676}$ of 36½℔, whole tare;

=1676 : 3650=2·177̇7℔ per cwt. Answer.

$\frac{100}{1676}$=·0596. $\frac{2{\cdot}1\dot{7}\dot{7}}{36{\cdot}5}$=·0596, proof.

When the numerator of a ratio is 100, the proof by ratios of article and estimate is little more than a repetition of the previous work; it is then best derived from ratios of article to article, and estimate to estimate.

Examples to be wrought, proved, and recited.

1. In 16 barrels of sugar, averaging, gross, 217℔ each; tare, 10℔ per barrel; what is the net weight?

2. In 9 *hhd.* of tobacco, averaging, gross weight, 5¾ cwt. *each; tare,* 76℔ per hhd., what is the net weight?

3. In 3 barrels of potash, weighing, gross, 155℔, $161\frac{7}{8}$℔, $149\frac{3}{5}$℔; tare, 10℔ per cwt., what is the net weight?

4. In 4 barrels of figs, weighing, gross, $76\frac{1}{2}$℔, $83\frac{1}{8}$℔, 66℔, $87\frac{5}{6}$℔; tare, 9℔ per cwt.; tret as usual; what is the net weight?

5. In 3 *hhd.* of sugar, weighing, gross, $\frac{1}{2}$ ton, Q $6\frac{3}{4}$, Q 8 $75\frac{1}{2}$℔; tare, 57℔ per hhd., what is the net weight?

6. In 2 *hhd.* of tobacco, weighing, gross, 12 *cw.* 16℔, $11\frac{1}{3}$ *cw.*; tare, 10℔ per cwt.; what is the net weight?

7. Two hhds. of sugar, weighing, gross, Q 9 1 *qr.* 17℔, Q 11 16℔, 5 *oz.*, are the property of two persons; what net weight is each person entitled to, allowing tare at 12℔ per cwt. English?

8. What is the net weight of 13 chests of sugar, weighing, gross, 13*cw.* $45\frac{3}{4}$℔; tare, 12℔ per cwt.?

9. In 3 chests of tea, gross weight, 192℔; tare, 6℔ per cwt.; tret as usual; what is the net weight?

10. What is the net weight of 3 frails of raisins; gross weight, 7*cw.* 13℔ 6*oz.*; tare, $21\frac{1}{2}$℔ per frail; tret as usual?

Purchase and Hire.

What is the maxim in purchase and hire?

MAXIM IN PURCHASE AND HIRE.

Purchase, rent, and hire, are estimated, according to rents and periods, by the rule of single proportion direct, and are proved by the same.

What is the usual process? — In purchase and hire, the article estimated is commonly a unit, in the same denomination with the article requiring estimate, as a pound, a week, a year; needing therefore multiplication only of the article requiring estimate, into the estimate given, by composites and aliquot or decimal parts. Most other cases are to be solved by means of a ratio.

In what sense do we multiply one composite into another? — The true multiplicand is in reality the only thing multiplied; the numbers of the other factor, although noted by composite marks to remind us whence they are derived, represent only turns and parts of a turn, or repetitions to be made of the true multiplicand.

APPLICATION.

At £3 9*s*. 8*d*., the cwt. English, what is the value of 15 packs of wool, weighing, net, Q 30 18℔ ?

14℔ $=\frac{1}{8}$ Q	£3 9*s*. 8*d*.		
	6×5 = Q30		9*s*. 0*d*. = £·45
			6*d*. = ·025
	20 18 0		2*d*. = $\cdot 008\dot{3}$
	5		
	104 10 0		9*s*. 8*d*. = $£\cdot 483\dot{3}$
3½℔ = ¼ of ⅛	8 $8\frac{1}{2}$		
½℔ = $\frac{1}{7}$ of $\frac{1}{32}$	2 $2\frac{1}{8}$		Q30 18℔ = 3378℔
	$3\frac{3}{4}$		
	£105 1 $2\frac{1}{4}$+ ans.		$\left\{\begin{array}{l}\frac{3\cdot 4833\dot{3}}{112}=\cdot 0311 \\ \frac{105\cdot 0593\dot{3}}{3378}=\cdot 0311\text{, proof.}\end{array}\right.$

The direction usually given for proving proportional operations, is to reverse the terms of the question; a direction seldom followed, because given in books that give also, what seems to supersede proof, solutions to the questions propounded. By reversing the terms is meant, that from the estimate found, we are to find the article requiring estimate: in the present case it would be as follows.

What quantity of wool may be purchased for £105 1*s*. 2¼*d*., at £3 9*s*. 8*d*., the cwt. English?

$$e\ s \text{ is the ratio, } \frac{£105\ 1s.\ 2\frac{1}{4}d.}{£3\ 9s.\ 8d.} \text{ of Q 1, } e\ g; = \frac{105\cdot 0593\dot{3}}{3\cdot 48333} =$$

$$3\cdot 48333 : 105\cdot 0593\dot{3} = \text{Q } 30\ 17\cdot 98℔ \text{ \&c. answer.}$$

The deficiency of the present answer is owing to the loss of fractions of a farthing in the former estimate found. Both methods of proof being here exhibited, a choice can be made; the method by equal ratios is most natural, generally least operose; by adopting it, we maintain a harmony and uniformity of operation in the most important branch of arithmetic, this of proportion; and it is always to be considered the standard proof, though, for brevity's sake, we may sometimes direct one mode of operation to be proved by another.

Examples to be wrought, proved, and recited.

1. At \$·11⅛ per ℔, what is the value of a barrel of sugar, weighing, gross, 317¾℔; tare, 7℔ per cwt.?

2. At \$1·03⅛ per ℔, what is the value of a chest of tea, weighing, gross, 67⅓℔; tare, 5½℔ per cwt.; tret, as usual?

3. At \$·10⅛ per quart, what is the value of a hhd. of wine?

4. At \$·01¼ per oz., what is the value of Q 1 3*qr*. 27℔ 5*oz*. of coffee?

5. At \$·93$\frac{3}{4}$ per ell English, how many yards of calico may be purchased for \$50 ?

6. If $\frac{4}{7}$ *yd.* of cloth cost \$·65, how many ells English may be purchased for \$65 ?

7. If an ounce of wrought silver be worth \$1·99$\frac{7}{8}$, what is the worth of a vessel weighing ℥ 1℔ 3 *oz.* 13 *dw.* 7 *gr.* ?

8. At £4·079 per yard, what is the value of 3 bales of cloth, each containing 6 pieces of 27 *yd.* each ?

9. At \$233 per annum, what portion of the rent is due for 7 *m.* 16 *da.* ?

10. At \$5$\frac{3}{4}$ per cwt., what is the value of 1213℔ 15 *oz.* of lead ?

11. At \$·35 per day for horse-keeping, what will be the expense during 6 *m.* 29 *da.* ?

12. If 17 ells of English cloth, $\frac{7}{8}$ *yd.* wide, cost £1 13*s.* 2$\frac{1}{4}$*d.*, what will 36 ells of similar cloth, 1 *yd.* 2*n.* wide, cost ?

13. If $\frac{1\frac{1}{3}}{1\frac{1}{4}}$ *oz.* of silver cost £$\frac{9}{4\frac{1}{2}}$, how much will $\frac{2\frac{7}{8}}{3}$ *oz.* cost ?

14. If $\frac{2}{3}$ of a ship be worth \$1763, what will $\frac{6}{7}$ of the remainder be worth ?

15. At £16 17*s.* per cwt., what is the value of $\frac{5}{9}$ *oz.* ?

16. If the value of $\frac{3}{4}$ of $\frac{5}{8}$ of a coal mine be \$2557, what is the value of the whole mine ?

17. If a man's wages be \$121 a year, how much is that by the day, reckoning no sabbaths ?

18. If 171℔ of soap cost \$13·117, what is the price per pound ?

19. At \$111 the acre, what is the value of 3 *rood* 15$\frac{1}{8}$ *yd.* ?

20. If 20 dozen of knives and forks cost, deducting discount, \$83·71$\frac{1}{4}$, what is the price per pair ?

21. At \$87 the year, what wages are due for 246 days ?

22. At \$·81 the bushel of corn, what is the value of 1199 gallons ?

23. At \$7·62$\frac{1}{2}$ per cwt. English of tobacco, what is the value of 4 *hhd.*, weighing, gross, 2719$\frac{1}{8}$℔, allowing tare at 6$\frac{3}{4}$℔ per cwt. ?

24. If 18 *cw.* 66℔ of steel cost \$191·27, what will 5 *cw.* 14℔ 11 *oz.* cost ?

25. At 12$\frac{1}{2}$ the cwt. of tobacco, how much may be purchased for \$·03 ?

26. At \$7$\frac{1}{8}$ the cwt. of sugar, how much may be purchased for \$·12$\frac{1}{2}$?

27. At \$9$\frac{1}{2}$ per ton of anthracite, what is the price per cwt. ?

28. At \$11$\frac{1}{2}$ per chaldron of bituminous coal, what is the price per bushel ?

29. At $11 per ton of anthracite, what is about the price per bushel? (See tables.)

N. B. Fractional numbers are as easily formed into ratios as integers or composites, but create a necessity of simplification.

Barter.

What is barter? — Barter is the exchange of one commodity for another, and is either simple or mixed.

In what does the difference consist? — Simple barter is of goods only, proportioned in quantity to their respective values in money; mixed barter is of goods equalized in value by the addition of money.

What must be known in such calculations? — In calculations of barter, the entire quantity of some one article, and the value or rate of all, must be known.

Why? — In barter, one value is set against another; and these cannot be equalized, unless we know the entire value on one side, and the rate on the other.

What will this entire value be in the statement? — The entire value of the quantity known is the article requiring estimate in barter transactions; for an equal value is sought in some other commodity.

By what rule is barter transacted? Barter is transacted by the rule of single proportion direct.

Under what maxim?

MAXIM IN BARTER.

In simple barter, the value of the quantity known is the article requiring estimate; in mixed barter, the difference in value is the balance payable in money.

Proof, equality of value on each side.

APPLICATION.

1. How much cotton, at $·12½ per lb, is equivalent to 5 *cw.* of sugar, at $·06¼ the lb?

$·06¼ × 500 = $31¼ value of sugar.

e s is the ratio,

$\frac{31\cdot25}{\cdot125}$ of 1 lb, cotton rated; $=\frac{6\cdot25}{\cdot025}=$ 250 lb of cotton. Answer.

250 lb
·12½
125
3000
$31·25, value of cotton. Proof.

2. Bartered ½ ton of sugar, at $·05⅜ the ℔, against 2 bales of cotton, of 4 *cw.* each, at $·11 the ℔; if unequal, what sum will compensate the difference?

$·11 × 800℔=$88 value of cotton.
$·05375×1000℔=$53·75 value of sugar.

$34·25 balance payable for cotton. Ans.

Proof, obvious.

Examples to be wrought, proved, and recited.

1. What quantity of tea, at $·92½ the ℔, must be given for 517℔ of coffee, at $·13?

2. How many bushels of wheat, at $2·11 the bushel, are equal to 116 gallons of rye, at $·98 the bushel?

3. Bartered, hops, at $36 the cwt., for iron, at $6⅓ the cwt.; what are the proportionate quantities of each?

4. Bartered, 3 *cw.* of cocoa, at $·8⅓ per ℔, for $10⅓ in cash, and the remaining value in rice, at $·04 the ℔; how much rice is needed to make up the deficiency?

5. If potatoes be sold at $·37½ the bushel, and canvass at $·13 the yard, how may these be proportioned in barter?

6. Bartered, 221 *yd.* of linen, at $1·12½, for 53 ells English of fine cloth; what was the price of the cloth per yard?

7. Bartered, 1 *cw.* 23℔ 6 *oz.* of butter, at $·11 the ℔, for 2⅔ *cw.* of sugar, at $·07 the ℔; how shall the transaction be equalized?

8. Bartered, 317 *ga.* of molasses, at $·26 the gallon, and 219℔ of sugar, at $·05¾ the ℔, for boards, at $13·83 the thousand square feet; deducting a payment of $50 in cash, how many feet of boards are required?

9. Bartered, West India molasses, at $·27 the gallon, for sugar-house molasses, at $·33 the gallon; how shall the quantities be proportioned?

10. Bartered, 153 *yd.* of cloth, at $3·17 per yard, for wax, at $16 the cwt., and $200 in cash; what quantity of wax is required?

11. Bartered, 17¼ *cw.* of tobacco, at $13 the cwt., for $150 in cash, and the balance in 80 bushels of corn; what was the corn per bushel?

12. Bartered, 211 bushels of rye, at $1·06 the bushel; $\frac{2}{9}$ of the value are to be paid in cash, $\frac{4}{9}$ in corn, at $·91 the bushel, and $\frac{1}{3}$ in wheat, at $2 the bushel; what is the sum, and what the respective quantities, sought?

13. Bartered, 4 barrels of sugar, averaging 3 *cw.* each, at $·07½ the ℔, for molasses at $·26¾ the gallon; what was the quantity of molasses?

Commission and Brokerage.

What is commission? — Commission is an allowance of so much per cent on the próceeds of goods intrusted to a person for sale.

What are próceeds? — Próceeds are the amount accruing from sales, gross proceeds being the entire amount; net proceeds, the amount after deducting charges.

What are the names given to parties engaged in this business? — The owner of the goods is usually called the principal, the seller is called the agent, factor, or commission merchant.

What is the meaning of agent? — Agent and factor are both from Latin words, signifying *to do, to transact.*

What is brokerage? — Brokerage is an allowance per cent to one who negotiates sales or purchases for another.

How do these two agencies differ? — A broker negotiates only; a commission merchant receives, stores, and delivers, goods.

How does this affect charges? — Commission proper is commonly a much higher rate than brokerage.

On what amount is rate estimated in mercantile transactions? — *Rate, in transactions of general merchandise, is the estimate on a hundred, and is the same with per-centage.*

In what manner is it calculated in any particular transaction? — From payments made to a principal, commission and brokerage are deducted; to charges made on a purchaser, brokerage is added.

Can you exemplify this? — Goods sold to the amount of $1000, on a commission of $5 per cwt., will realize $950 only to the principal; brokerage of $1 per cent, on the same amount, charged to a purchaser, will increase the bill of sales to $1010.

Can you generalize this transaction? — Commission and brokerage are deducted, at so much per cent, from the account of sales; brokerage is added, at so much per cent, to the amount of an invoice.

When the estimate sought is at so much per cent, what are *a e* and *e g*? — In most cases of a per-centage sought, the article estimated, as the terms import, is 100; the estimate

given is the rate itself, since from the rate of a hundred we are to estimate any other sum.

Discount proper is an exception to the estimation from 100.

When the estimate given is a per-centage, how may we distinguish the rate of a unit? — ***The term, rate of unity, designates the hundredth part of a per-centage, and is equivalent to the division of an estimate given, by* 100*, the article estimated.***

Having the rate of unity, how may you find the estimate sought? — The article requiring estimate, multiplied into the rate of unity, produces the estimate sought; for the article estimated is a divisor, and the rate of unity is its quotient.

Can you exemplify this? — If a commission be \$5 per cent, the commission on a single dollar is \$·05; nów \$·05 are decimals of the second place; that is, 5, divided by 100.

$\frac{1}{100}$ of \$5, rate; =\$·05, rate of unity.

It is of great importance that the learner should recollect the origin and import of this term, *rate of unity.*

How are computations made in these branches of business? — Commission and brokerage are estimated by the rule of single proportion direct.

Under what maxims?

MAXIMS IN COMMISSION AND BROKERAGE.

Commission and brokerage sought are products of the amount of sales multiplied into a hundredth part of the rate.

Proof, by division.

Net proceeds, under agency, are the amount of sales, minus the charges and commission.

Broker's bill is the amount of invoice, plus the brokerage.

Proof, of decimal operation, by aliquot parts; and conversely.

APPLICATION.

1. Account of sales, \$565·19; commission, \$7 per cent; what are the net proceeds?

$\left\{ \begin{array}{c} e\_s \text{ is the ratio, } \frac{565·19}{100} \text{ of \$7;} \\ =565·19\times·07 \end{array} \right\}$

\$565·19 × ·07 rate of unity.
39·5633 deduct commiss.

\$525·6267 net p. Answer.

The best proof of calculations so simple, is a formal statement, that we may deliberately consider whether the factors have been *rightly* taken.

2. Broker's invoice for wines, $\$983{\cdot}661$; brokerage, $\$\frac{7}{8}$ per cent; what was the bill?

$\cdot 00\frac{7}{8} = \cdot 00875$

$983·661 invoice.
·00875 rate of unity.

4918305
6885627
7869288

8·60703375 brokerage.
983·661 add invoice.

$992·268 answer.

$$\left\{\begin{array}{l} \$\cdot 00\frac{4}{8} = \$\cdot 00\frac{1}{2} \\ \cdot 00\frac{2}{8} = \cdot 00\frac{1}{4} \\ \cdot 00\frac{1}{8} = \cdot 00\frac{1}{8} \end{array}\right.$$

$983·661 invoice.
4·918305
2·4591525
1·22957625

$992·26803375 proof.

Examples to be wrought, proved, and recited.

1. What is the commission on $\$1463{\cdot}06\frac{1}{4}$, at $\$4\frac{3}{4}$ per cent?
2. What is the brokerage on $739·968, at $\$\frac{1}{3}$ per cent?
3. Account of sales, £381 16*s.* $4\frac{1}{2}$*d.*; commission, £$3\frac{1}{2}$ per cent; what are the net proceeds?
4. Invoice, £5061 9*s.* 2*d.*; brokerage, £$\frac{5}{8}$ per cent; what is the bill?
5. Commission, $\$6\frac{1}{4}$ per cent; account of sales, $\$2617{\cdot}43\frac{7}{8}$; what are the net proceeds?
6. A broker receives $9683, to make purchases therewith; deducting a per-centage of $\$1\frac{3}{8}$, what remains to be laid out?
7. An agent receives £1000, for the purchase of goods, which he resold for £1266; at a commission each way of £$1\frac{1}{2}$ per cent, what are the net proceeds?
8. Net proceeds $617·15; commission, $33; what was the rate per cent?
9. Commission on proceeds, $56; rate, $\$4\frac{3}{4}$ per cent; what was the amount of sales?

Trade Discount.

What is trade discount? — Trade discount is an allowance per cent on the invoice prices of goods sold to a dealer, with or without reference to time.

What usually obtains the largest discount? — Present payment is commonly most favored.

How does trade differ from discount proper? — Discount proper is an allowance limited by law, and made in transactions entirely pecuniary; trade discount is a deduction from the price of goods, and altogether unlimited.

Under what maxim are the estimates made ?

MAXIM IN TRADE DISCOUNT.

Trade discount sought is the product of the invoiced amount multiplied into one hundredth part of the rate.

Net charge is the amount of invoice, minus the discount.

Proof, of decimal operation by aliquot parts, and conversely.

APPLICATION.

What is the trade discount on an invoice of £316 5*s*. 9*d*., at £12⅝ per cent ?

5*s*. = £·25	£40	= £40		
6*d*. = ·025	·55	=	11*s*.	
3*d*. = ·0125	·025	=		6*d*.
	·015	=		3½
5*s*. 9*d*. = £·2875	£40·590	= £40 11*s*. 9½*d*. ans.		

e s is the ratio,

$\frac{316 \cdot 2875}{100}$ of £12⅝ discount per cent ; =316·2875

12⅝

6 : 15814375

2635729⅙
37954500

£40·590229⅙

Examples to be wrought, proved, and recited.

1. Goods invoiced at $76·191 ; what is the discount, at $8⅞ per cent ?
2. Invoice, $399·65¼ ; discounting at the rate of $23 per cent, what is the net charge ?
3. What is the discount, and what due, on £575 17*s*., the invoice, allowing £7 per cent for prompt payment ?
4. What sum is now payable on $1011·99⅞, invoice amount due 6 months hence, allowing a discount of $3½ for cash ?

Insurance.

Before he enters upon insurance, the learner is recommended to proceed to *Imposts and Stocks*, as those subjects involve no principle which has not already been explained; to have introduced them here would have interrupted the series of commercial rules; and to defer the consideration of them till he shall have reached them in the order of the book, will render them no exercise.

What is insurance? — Insurance is a contract of limited indemnification, contingent on losses at sea, or by fire, and granted in consideration of a certain per-centage, called the premium, paid on the whole sum insured.

What is the advantage of insurance? — Insurance is of the highest advantage to individuals, by saving them from the ruinous consequences of losses; and to the public, by lessening the risk with which commerce is carried on.

Is insurance always of advantage? — Since there must be a profit on insurance, or no one would undertake it, should a person's property at sea, or on land, be so extensive, that the amount of insurance would probably exceed the amount of his losses, insurance would, as probably, be of no benefit to him.

Does any mischief arise from insurance? — It is believed, that destruction of property by fire is increased, through carelessness and wilfulness, in consequence of the facilities to insure.

Are there any means of correcting this evil? — The remedy, and the duty, is with the insurers; thoroughly to investigate every suspicious case that comes before them for indemnification.

How are the parties in these transactions named? — They who grant insurance are called insurers, or underwriters; they who take it are the insured.

Does any deed intervene? — A policy is the deed of insurance.

How is insurance regulated? — The premium is proportioned to hazard, time, and sum insured.

Is time an element of the calculation? — Insurance being commonly for a year, season, voyage, or trip, and considered in the premium, the introduction of time, as a factor, we presume to be unnecessary.

Is the indemnification against all losses? — A certain average loss, or loss not exceeding a small and fixed per-centage, is excepted; or losses would be perpetual, and premiums much increased.

19

What is general average? — General average is a proportioning of loss, purposely incurred for the safety of ship and cargo, among persons interested in either; it is found by the rule of distribution.

To what extent may one insure? — Insurance may be made to the full extent of one's interest; nothing beyond can be recovered by law.

Whence the prohibition? — Dishonest men have taken advantage of an unlimited power to insure, purposely to incur risks that shall entail damage far below the amount insured; in short, to create losses.

Are such offences penally visited? — Wilfully to destroy property, with a view to defraud insurance offices, has been made a capital felony.

What are the subjects of maritime insurance? — Maritime insurance comprehends ship, tackle, cargo, freight, and the premium itself.

Why this? — When a loss has been incurred, from the indemnity is deducted the premium; this may render the insurance but partial; to perfect it, therefore, the amount of premium may be added to the principal sum insured, and premium be paid on premium.

Arithmetically, in what manner? — To cover the entire interest in case of loss, a hundred is insured for every time that 100, less the premium, is contained in the principal sum; and in the same proportion for any part of a hundred.

How is the indemnity thus equalized with the loss? — This precisely compensates the deduction of premium; for since only the hundred, short of its premium, is paid, on settling for a loss, if every such short hundred contained in the principal be insured as a perfect hundred, the indemnification is complete.

What will be the arithmetical process? — To cover the premium, the entire principal must be increased proportionally to the increase of a hundred, less the premium, to an exact hundred.

Then how is the proportion formed? — Since 100, less the premium, is estimated, in the calculation of complete insurance, at a full hundred; 100, less the premium, becomes an article estimated; and 100, the estimate given; the principal sum being the article requiring estimate.

What stage of the insurance is this? — This is a prior step in the calculation of insurance, taken to determine what amount shall be insured.

By what rule are the calculations governed? — Calculations of insurance are governed by the rule of single proportion direct.

What are the maxims?

MAXIMS IN INSURANCE.

Insurance sought is the product of the sum insured multiplied into a hundredth part of the premium. *Proof*, by division.

Amount to be insured, covering the premium, is proportioned as one hundred, less the premium, to an exact hundred. *Proof*, as in proportion direct.

APPLICATION.

1. What is the whole premium, annually to be paid, on house insurance, valued at $4000, rated at $¼ per cent; on goods, valued at $3025, rated at $½ per cent?

$4000 house.	$3025 goods.	$10 house p.
·00¼ rate of unity.	·00½ rate of unity.	15·125 goods p.
$10·00	$15·125	$25·125 answer.

Proof, obvious.

2. What sum must be insured, to cover the entire interest on freight, valued at $2100; and what the insurance, at a premium of $3½ per cent?

e s is the ratio,

$\frac{2100}{100-3\cdot5}$ of $100, amount proportioned; $=\cdot965 : 2100$ $=$ $2176·165803, amount to be insured. Ans.

$\left\{ \begin{array}{l} \frac{100}{96\cdot5}=1\cdot0362. \\ \frac{2176\cdot1658}{2100}=1\cdot0362 \text{ proof of amount.} \end{array} \right.$

$2176·1658 principal.
·07×·5 rate of unity.

152·331606
·5

$76·1658030 insurance. Ans.

The insurance is proved by its being the exact difference between the principal sum and the amount covering premium; indeed, the operations prove one another.

Examples to be wrought, proved, and recited.

1. What is the cost of insurance, for a year, on a house valued at \$2100, at a premium of \$$1\frac{1}{4}$ per cent; on goods, valued at \$1561, at \$$1\frac{3}{4}$ per cent?

2. What is the amount of premium to be paid on merchandise, valued at \$7000, at the rate of \$$4\frac{3}{5}$ per cent?

3. What is the insurance on ship and cargo; ship valued at \$5101, at a premium of \$$1\frac{3}{8}$ per cent; goods valued at \$6093, at a premium of \$$2\frac{1}{6}$ per cent?

4. On a total loss, to the amount insured of \$7394, premium \$$4\frac{10}{11}$ per cent, how much have the insurers to pay?

5. What sum will cover the interest on merchandise, valued at \$17009; and what the insurance, premium being \$3·6 per cent?

6. What sum will cover the interest in a policy including ship and cargo; ship valued at \$9783, premium, \$$6\frac{5}{12}$; goods valued at \$8076, premium, \$$5\frac{4}{5}$ per cent?

7. A total loss having occurred on merchandise valued at \$1181, premium covered, \$$3\frac{1}{3}$ per cent, how much more will be received, than if the premium had not been covered?

8. Insurance is to be made on merchandise, valued at \$10000; at a premium of \$4 per cent, what will be the different expense of insuring or not insuring the premium?

9. What is the insurance on \$6044, at \$1·75 per cent premium, covering interest entire?

10. Proposing to insure \$5167, on merchandise, the premium demanded on which is \$$6\frac{3}{7}$ per cent, what will be the amount of disadvantage, in case of a total loss, of not having insured the premium?

Profit and Loss.

What are the commercial terms of purchase, advantage, and disadvantage? — Terms of negotiation, frequent in commerce, are, cost, profit, loss, sale price, credit price, cash price.

What is the first? — Cost is the price paid for any thing; in commerce it usually denotes the price paid by merchant or dealer.

The second? — Profit is the excess of price above cost, accruing to a seller.

Its opposite? — Loss is the deficiency of price below cost, sustained by a seller.

The fourth? — Sale price, or próceeds, is the money *actually obtained* for goods, including profit, or falling short of *cost.*

The fifth? — Credit price is the charge made in reference to a distant payment.

Its opposite? — Cash price is the lowest charge on prompt payment.

On what grounds is the claim to profit founded? — Profit is founded on the claim of a dealer to compensation for the employment of capital, for labor bestowed, for attention given, and for talent exercised, to public advantage, in obtaining useful and elegant commodities.

Is there any more extensive ground? — There is also the general right of a man, to ask what he will for his own.

A right without limitation? — Whatever limitation the law of society may put to the right of property, that right always extends to the demanding of a just and fair price.

What is a fair price? — The market price of the day, when it has not been raised by craft and combination, is the just and fair price of any thing.

Is it unfair to ask more? — It may often be perfectly fair to ask more than the market price; since it is very possible there may be no wish to sell, though the purpose might be changed if an extraordinary price could be gotten; but this is a case entirely different from the routine of ordinary business, the sole object and end of which is, to buy and sell.

What, under the circumstances of general trade, may render conduct reprehensible? — Whenever advantage is taken of the ignorance or distress of a purchaser, to obtain from him more than the just and market price, the conduct of the seller is unfair.

Is a dealer bound to tell all he knows? — A dealer is *not* bound to inform a purchaser of all he knows; he is bound, like every other man, to truth and candor in his representations; but a merchant's knowledge is his property, by which he must maintain himself and family; and it would be impracticable to carry on business upon any other principle.

Is there not an adage on this subject? — *Let the buyer look to it,* is an old legal maxim, which only fanaticism will impugn.

Is there none of still higher authority? — To do unto others as we would be done unto, is a maxim of divine authority; the buyer, therefore, who would deprive a merchant of his just profit is no less a transgressor of this precept, than the merchant who puts off damaged for sound goods.

What are the chief occasions of loss? — Beside default in payment, loss is chiefly occasioned by a diminution of demand, an excess of supply, a deterioration of article.

In accounts, what is understood by profit and loss? — Profit and loss is an application of the rule of single proportion direct to gains and losses in trade, either realized or in prospect.

What are the cases requiring an estimation of profit? — The occasion for the adjusting of profit is that of determining a sale price.

How is it conducted? — If the rate of profit be fixed at so much per cent on the cost, it must be proportioned to every unit of the cost, or to every single article for sale; if profit in a round sum be determined upon, that sum will be the estimate given for an apportionment among the hundreds and units of the cóst, the entire cost being the article estimated; for in it is to be realized the entire profit.

What is the process? — Profit, at a rate per cent on the cost, is apportioned by multiplying the cost entire, or any specific part of it, into the rate of unity; their product is the profit on the cost multiplied, and the sum total of cost and profit is the sale price, either of the entire article, or of a part.

Can you exemplify this? — If the cost of an entire article be $100, and the profit rated on the cost be $20 per cent; then, for every five dollars of cost, the sale price will be $6; for five times $·20, the rate of unity, is equal to a dollar; this added to the $5 cost, makes $6 sale price.

What other occasion is there for the estimation of profit? — One principal occasion for the estimation of profit is the ascertaining of it on próceeds, realized or in prospect.

What is the manner of this estimation? — Profit being the sale price, minus the cost, when the cost is known, the profit is known on subtraction of the one from the other.

Suppose it not to be known? — To determine the amount of profit, if the cost be not known, the rate of profit must be known; if this have been computed on the cost at a per-centage, then 100, plus the per-centage, will be an article estimated, whence the profit on any amount of próceeds may be learned.

Why must it be 100, plus? — Próceeds include both cost and profit; when therefore profit has been adjusted on the cost, the proceeds' hundred will not include an entire per-centage of the profit, for the remainder is less than the cost hundred; but on every hundred of the proceeds, plus the per-centage, an entire per-centage of profit is made.

Is there never occasion to ascertain entire profit from the cost itself? — The ascertaining of profit from cost is conducted precisely as the adjusting of it for a sale price.

Is profit always thus ascertained? — In some branches of business, the sale price being previously fixed, the computation of profit is made between parties concerned, by deduction from the sale price.

What is the effect of this? — Where profits are computed by deduction from the sale price, then, on every hundred of the proceeds, an entire rate per cent has been made; consequently, in ascertaining profits, 100 is the article estimated.

Can you exemplify this? — If profit allowed by deduction be $25 per cent, then in every hundred of the proceeds, there will be $75 cost and $25 profit.

Is this in reality a profit per cent? — Both methods of estimating profit are spoken of precisely in the same terms, although they differ in their results exceedingly; profit computed by deduction being necessarily on a principal sum if less than a hundred; one is a rate on cost; the other a rate on receipts.

What are the cases of loss? — Cases of the computation of loss are the ascertaining of actual loss from actual proceeds, and the fixing of a sale price at a loss calculated upon.

When is this done? — The deterioration of articles, and reduction of demand, sometimes make it necessary to sell commodities at a loss.

How is the estimate formed? — Loss, at a rate per cent, is apportioned to cost, by multiplying the cost entire, or any part of it, into the rate of unity; the product is the loss proposed on the amount of cost multiplied; and the cost, minus the loss, is the reduced sale price.

How is loss ascertained from the proceeds? — Loss being the deficiency of the proceeds from the cost, if the cost be known, the loss is known, by subtraction, of the one from the other.

Suppose it not to be known? — Loss, to be computed on proceeds, at a rate per cent, is proportioned as 100, minus the loss per cent, to the per-centage itself, the estimate given.

Why minus? — Proceeds, in case of loss, fall short of the cost; consequently, for every hundred of the cost, less than a hundred has been received, and the whole loss on a hundred sustained; the article estimated therefore must be less than a hundred.

Can you exemplify the case of loss? — If the cost price of an article be five dollars, and a loss be calculated upon of $20 per cent, the sale price of the article will be reduced to $4; for 5 times $·20, the rate of unity, is $1; this, subtracted from $5 cost, leaves $4 for the sale price.

Can you exemplify the estimation of loss on proceeds?— If proceeds be $4, at a loss of $20 per cent, the article estimated is $80; for $80 only are received for what cost $100; and the actual loss, on the receipt of $4, is $1, as shown by the proportion. $\frac{4}{100-20}$ of $20, loss per cent; =$1.

What are the maxims under this head?

MAXIMS IN PROFIT AND LOSS.

Profit and loss, at a rate per cent, to be apportioned to cost, or estimated therefrom, is the product of cost multiplied into a hundredth part of the rate.

Profit, to be computed on proceeds, at a rate per cent on the cost, is proportioned as 100, plus the rate, to the per-centage itself, the estimate given.

Profit, to be computed on proceeds, at a rate per cent in reduction of the sale price, is proportioned as 100 to the per-centage.

Loss, to be computed on proceeds, at a rate per cent, is proportioned as 100, minus the rate, to the per-centage itself, the estimate given.

Sale price is the cost, plus the profit or minus the loss.

Proof, as in proportion direct.

APPLICATION.

1. Purchased, 128 *yd.* of cloth for $360; at what rate per yard must it be sold, to bring a profit of $20 per cent?

$360	entire cost.	$\frac{1}{128}$ of $432	=	$3·375	answer.
·20	rate of unity.	384		128	
72·00	entire profit.	480		27000	
360	add cost.	384		40500	
$432	entire sale price.	960		432·000	proof.
		896			
		640			
		640			

To fix a retail price, we must, of course, first ascertain the entire sale price. Let it be noticed, that all the parts of this operation are proportional, the proof being a kind of reversal; ($\frac{128}{1}$ of $3·375); and that the formal process, though bearing the semblance of division, is, properly, *fractional multiplication.*

2. Sold, 128 *yd.* of cloth, at \$3·375 per yd.; entire cost being \$360, what was the gain, entire and per cent?

\$3·375 retail price. $\frac{100}{360}$ of \$72, entire gain; =3·6 : 72= [\$20 per cent. Ans.
128 *yd.*

27000
40500

432·000 proceeds
360 deduct cost.

\$ 72 entire profit. Ans.

$\frac{20}{72}\times\frac{5}{5}=\frac{100}{360}$, proof.

3. Sold, 128 *yd.* of cloth, at \$2·25 per yard, and at a loss of \$20 per cent, what was the entire loss, and the whole cost?

\$2·25 sale price.
128 *yd.*

1800
2700

\$288·00 proceeds.

$\frac{288}{100-20}$ of \$20, loss per cent; $=\frac{288\times20}{80}$=8 : 576=\$72 entire loss.
288 proceeds.

\$360 whole cost. [Ans.

$\frac{20}{80}=\frac{2}{8}$. $\frac{72}{288}=\frac{2}{8}$, proof.

4. Sold, 128 *yd.* of cloth, at \$3·375 per yard, and a profit of \$20 per cent; what was the entire profit, and the whole cost?

\$3·375 sale price.
128 *yd.*

27000
40500

\$432·000 proceeds.

$\frac{432}{100+20}$ of \$20, gain per cent; $\frac{432\times20}{120}$=12 : 864=\$72, entire gain.
\$432 proceeds. [Ans.
72 deduct profit.

\$360 cost. Ans.

$\left\{\begin{array}{l}\frac{20}{120}=\frac{1}{6}\\ \frac{72}{432}=\frac{1}{6}\end{array}\right.$, proof.

5. Cost of an article, \$·18 ; sale price, \$·31 ; what is the profit on a hundred articles, and what the profit per cent ?

$\$\overline{\cdot 31 - \cdot 18} \times 100 = \$13$, profit on 100 art. Answer.

$\frac{100}{\cdot 31}$ of \$·13, profit rated; $=31 : 1300 = \$41{\cdot}935$, gain per [cent. Ans.

$\frac{100}{\cdot 31} = 322{\cdot}5$. $\frac{41{\cdot}935}{\cdot 13} = 322{\cdot}5$, proof.

Examples to be wrought, proved, and recited.

1. Purchased, 316$\frac{1}{2}$℔ of Havana coffee, at \$·11$\frac{1}{2}$ per ℔; if resold at \$5$\frac{1}{2}$ per cent profit, what will be the proceeds ?

2. Sold, 3 tons English of pearl ashes, at \$117$\frac{1}{5}$ per ton, by which was made \$23 profit; at what rate per cent ?

3. Purchased, 5 casks of Malaga raisins, at \$8$\frac{2}{3}$ the cask; if resold at \$9$\frac{1}{7}$, what will be the proceeds and the profit per cent ?

4. Sold, 77 *yd.* of Company's Nankin, at \$·61 per yard, on which a profit was made of \$15 per cent; what was the cost, per yard and entire ?

5. Purchased, 693$\frac{2}{9}$℔ of James River tobacco, at \$·04$\frac{1}{4}$ per ℔ ; with a profit of \$7$\frac{1}{8}$ per cent, what will be the sale price per ℔, and what the proceeds entire ?

6. Sold, Q $\frac{1}{2}$ of Hyson tea, at \$·91$\frac{3}{8}$ per ℔, on a profit of \$13$\frac{3}{5}$ per cent; what was the entire gain and the cost ?

7. Purchased, Q 1$\frac{1}{2}$ of English white lead, at \$12$\frac{1}{2}$ per ℔; at what rate per ℔ must it be sold, to make a profit of \$11$\frac{1}{9}$ per cent; and what will be the entire proceeds ?

8. Sold, 83℔ of deer skin, at \$·14$\frac{1}{2}$ per ℔, by which a loss was incurred of \$3$\frac{1}{6}$ per cent; what was the cost ?

9. Sold, 336℔ of American wool, at \$47$\frac{2}{3}$ the ℔, on a profit of \$5$\frac{1}{5}$ per cent; what were the entire profit and proceeds ?

10. Purchased, 2 pipes of Malaga wine, at \$56·7 the pipe; at what rate per gallon will a profit of \$26 be made on the whole, and what will be the profit per cent ?

11. Sold, 305 pair of India rubber shoes, at \$1·56$\frac{1}{4}$ the pair, on a profit of \$28 per cent; what was the entire profit and cost ?

12. Purchased, $\frac{1}{2}$ a ton of Missouri pig lead, at \$·03$\frac{7}{8}$ the ℔; at what rate must it be sold, to make a profit of \$7$\frac{5}{8}$ per cent, and what will be the proceeds ?

13. Sold, 619℔ of damaged leather, at a loss of \$15$\frac{3}{11}$ per cent; entire cost having been \$148·56, what was the sale price per ℔, and what the proceeds ?

14. Sold, 323℔ of hops, on a profit of \$13$\frac{2}{7}$ per cent; entire cost being \$116·28, what was the retail price, and what the proceeds ?

15. Laid out, \$123 in rice, at \$·03½ per lb, and sold it at a loss of \$10 per cent; what was the retail price, and what the proceeds?

16. Purchased, 27 barrels of prime beef, at \$10½ per barrel; and sold it, in consequence of injury sustained, at a loss of \$37⅚ per cent; what was the sale price per barrel, and the proceeds?

17. Laid out, \$1700 in furs, weighing 1265¾lb; if sold at an average of \$1·83 per lb, what will be the gain, per cent and entire?

18. At \$·11$\frac{5}{9}$ profit on \$1·37½, what is the gain per cent?

19. At \$18 cost per ton for grindstones, what will be the gain or loss, entire and per cent, if sold at \$·99 the cwt. English?

20. Purchased, 1 *hhd.* of Lisbon wine, at \$1 the gallon; what will be the gain or loss, entire and per cent, if retailed at \$·4 the gill?

21. Purchased, sugar, to the value of \$413; what profit per cent will make the proceeds \$496·9?

22. Purchased, 129 gallons of sperm oil, at \$·92⅞ per gallon, and sold it at \$1·13 per gallon; what was the gain per cent, and entire?

Distribution.

What is distribution in accounts? — Distribution is an application of the rule of proportion direct to the division of bankrupts' effects among creditors, of testators' effects among legatees, in the case of deficient assets, and other cases; to the apportionment of losses under the head of general average, or losses purposely incurred for the safety of ship and cargo; and of salvage.

By what name is distribution usually known? — Distribution and partnership both go under the name of fellowship; but they are naturally separated by differences of origin, of purpose, and of computation.

Can you explain these differences? — Associates in distribution are brought together accidentally, pursue no common advantage, and make no claims in respect of priority in time.

What are the principles of distribution? — Equal claims entitle to equal shares, equal shares impose equal burdens.

What of unequal? — Unequal claims and shares participate in the ratio of their sum total to each.

How is this demonstrated? — Property to be distributed, losses to be sustained, and contribution to be levied, must be apportioned among the several claimants and obligees; the

amount consequently of all their claims and shares is the article estimated, at the whole sum to be distributed or levied; and each particular share, severally, is an article requiring estimate.

Why should this be so? — Because it is just, that he who has the largest claim should receive the largest share of what remains to satisfy it; and that he who has had most at risk should contribute most to the cost of a common preservation.

Under what maxim then is distribution made?

MAXIM IN DISTRIBUTION.

Claimants and obligees participate severally in the ratio of the sum total of their sharés to each share in particular.

Proof. Equality of the portions found with the entire sum to be apportioned.

A distinction is not here made between equal and unequal shares, because the maxim comprehends both, though the just manner of distribution in the former case be self-evident.

APPLICATION.

1. The effects of a bankrupt, amounting to \$600, are to be distributed between A, to whom he owes \$515, B, a creditor for \$98, and C, a creditor for \$470; how much should each receive?

\$515 A's claim	A's portion is the ratio, $\frac{515}{1083}$ of \$600,
98 B's	sum of the portions; $=10{\cdot}83 : 3090=$
470 C's	\$285·318$\frac{606}{1083}$
	B's, $\frac{98}{1083}$ of \$600; $=10{\cdot}83 : 588{\cdot}0=$
a e=\$1083 sum total of	\$54·293$\frac{681}{1083}$
[the claims	C's, $\frac{470}{1083}$ of \$600; $=10{\cdot}83 : 2820=$
	\$260·387$\frac{879}{1083}$

285·318$\frac{606}{1083}$	A's portion	answer.
54·293$\frac{681}{1083}$	B's	
260·387$\frac{879}{1083}$	C's	
\$600·000	proof.	

2. Of a cargo of wine, consisting of 100 pipes and 200 *hhd.*, 20 *p.* and 50 *hhd.* were started, or thrown overboard, to lighten the vessel; how much must three owners, one of 40 *p.*, one of 150 *hhd.*, and a third, the owner of the remainder, bear, each, of the loss; the three being equal partners of the vessel?

150*hhd.*=75*p.* 50*hhd.*=25*p.* 20*p.*+50*hhd.*=45*p.* entire loss.

40*p.* 1st share
75*p.* 2d
85*p.* 3d

a e 200*p.* sum total of shares.

1st contribution is the ratio, $\frac{40}{200}$ of 45*p.*, loss; =2,0 : 18,0=9*p.*
2d, $\frac{75}{200}$ of 45*p.*; $=\frac{75\times9}{40}$=8 : 135=16⅞*p.* } Ans.
3d, $\frac{85}{200}$ of 45*p.*; $=\frac{85\times9}{40}$=8 : 153=19⅛*p.*

45*p.* proof.

The three being supposed equal partners of the vessel, makes it unnecessary to introduce its value, as a subject of contribution.

Examples to be wrought, proved, and recited.

1. Insurance having been effected, to the extent of $2000, on ship and tackle, valued at $3555, and lost at sea, what will the respective owners of $\frac{3}{8}$, of $\frac{1}{3}$ of the remaining $\frac{5}{8}$, and of the remainder after that, be entitled, severally, to receive from the insurers?

2. The effects of a bankrupt amount to $3120·387; he owes, to 3 creditors, $500 each; to 2 creditors, $750 each; and to 4 creditors, $1000 each; how much is each creditor entitled to receive out of the estate?

3. If A receive one part of 339\frac{3}{8}$; B, two parts; C, three parts; how much will each receive?

4. The property of a deceased person amounting to $618·06¼, his debts were, to one, $230·64; to a second, $318·57; to a third, $200; what will each receive?

5. The legacies of a person deceased amount to $40500, but his effects realize only $31180·13½; what will be the distribution among the legatees, two of $10000 each, one of $5000, and three of the remainder among them?

6. A testator's effects fall short, after his decease, $\frac{1}{13}$ of his own estimation; but he has directed that $500 shall be paid to A without any deduction, in case of deficiency of assets; what will be the deduction from the remaining legacies, to B, of $1000; to C, of $763?

7. Should a testator direct his property of \$6000 to be divided in the following proportions, namely, $\frac{3}{11}$ to one person, $\frac{4}{13}$ to a second, $\frac{1}{13}$ to a third, appointing a fourth residuary legatee; how much would each receive out of the \$6000?

8. In a case of general average, the jetson was estimated at \$1183·28; what will be the apportionment of loss among the owners of cargo, respectively, to the amount of \$616, of \$1080, of \$2516, and of ship, valued at \$5170?

Partnership.

What is partnership? — Partnership is of a few persons, who unite their skill and their capital, for the carrying on of a particular business, at the risk of all they possess; or is between the members of joint stock companies, at the risk only of the amount of their respective shares.

Is there none still more limited? — There are also partnerships in particular adventures, subject to the same liabilities, to the extent of the adventure, as general partnerships.

Limited responsibility partnerships, as they might be termed, begin now to be introduced. In the State of New York, by giving public notice, and making a public record, a partner may limit his responsibility to the sum in said notice and record specified.

What are the advantages, and the contrary, arising from partnership? — Private partnerships enjoy the customary advantages arising from union; but are subject to the casualty of utter ruin through the folly or misconduct of a single partner, all being answerable for the acts of any one, done in the business of the coparceny.

What of the more public kind? — Joint stock institutions, for the conducting of business which, in its own nature, requires more capital than even rich men usually possess, as the banking business, insurance, manufactures requiring expensive machinery, and novel inventions of great risk, may be of public benefit.

Are they ever disadvantageous? — Companies of scheming men, associated merely for the purpose of taking advantage of a popular feeling to sell shares at an advance, are public robbers.

Is there no other case of mischief? — When instituted for the conducting of business ordinarily made a pursuit, and the means of living, by individuals and families, such companies are in the nature of a monopoly, and a nuisance.

What are the questions arising in partnership? — The arithmetical questions in partnership are of profit and loss, and division of stock.

How are they determined? — All questions of account in partnership are determined by proportion direct.

What variety is there in the terms of partnership? — Partnership is either on equal or unequal terms; the difference arising from difference of skill, labor, capital, and time during which capital has been employed.

Demonstration of rule.

What belongs to the rule? — Capital, and the time it has been in use, are the proper subjects of the rule of partnership, so far as they affect the distribution of profit and loss.

How is this distribution proportioned? — Equal shares during equal times entitle to an equal distribution; inequality of shares or times renders claims to distribution unequal.

Why is time introduced? — Capital earns profit, for the most part, in proportion to the time it is employed; therefore, the longer the time, the larger any portion of profit made; and, unless limited by special circumstances, of loss sustained, if the trade be unfavorable.

What is the meaning of stock? — Stock, in partnership, signifies the whole capital employed, whether laid out in goods, or remaining in cash.

With a single inequality, how is distribution made? — Unequal shares participate of profit and loss in the ratio of the stock, or sum total of the shares, to each particular share; for when the times only are equal, a man's proportion of gain or loss must be his proportion of the stock.

Is this universally the case? — The general practice is now, not to proportion profit, but to allow interest to a partner for any excess of capital that may pertain to him: this secures him, to a certain degree, from loss.

What is the effect of unequal time? — Unequal times participate of profit and loss in the ratio of the sum total of the times to each partner's time; for when the shares only are equal, a man's proportion of gain or loss must be the proportion of time during which his share has been employed.

Can you exemplify this? — If the shares of two partners be each \$100, their profits, \$30, while one has been engaged in the business 12 months, the other 6 months only, the share of the former will be \$20 out of the \$30; for having been engaged double the length of time, he is entitled to double the profit.

How do you apply the example to the maxim? — The sum of the times is $12m.+6m.=18m.$; and the ratio, $\frac{12}{18}=\frac{2}{3}$; and $\frac{2}{3}$ of $30, the entire profit, is equal to $20, the share assignable to the longest period.

Why should not the ratio be, as the longest period? — The *ratio* cannot be as the longest period; for thus the partner who had been longest in the concern would sweep away all the profit, or be subject to all the loss; since the time estimated and requiring estimate would, in his case, be equal.

Why should it be, as the sum of the times? — Times, in arithmetic, signify, or indicate, turns, each occupying a portion of time, limited by the nature of the thing and subdivided at the will of the accountant; consequently a greater length of time gives a greater number of turns; and all the subdivisions of time answer to the whole number of turns.

Can there be a simultaneity of times? — All time, that is not successive, is one and the same, in whole or in part; but the turning of money in different shares, at the same time, is without limit.

What is the third case? — Shares and times, both unequal, participate in the ratio of the sum of the products of every share multiplied into its time, to the separate product of each share and time.

How is this demonstrated? — The whole gain or loss has arisen on the sum of the shares employed during a certain time; but all the shares have not been employed all the time, in the case of a double inequality; therefore the article estimated cannot be the product of the stock multiplied into all the times or turns; but is the entire product of all the shares, separately multiplied, each into its respective time; for thus has the gain or loss accrued.

Why to the separate product of each share and time? — The entire profit or loss being the product of all the shares, multiplied each into its time, or being so estimated in calculation, the component parts of the profit or loss correspond, of course, each with a certain separate product of share and time.

After all, how is gain the product of share multiplied into time? — That gain is the product of capital and time is manifest to everybody; but the time, however short or long, is, in reality, a unit, with reference to the advantage obtained through its intervention; its multiplication therefore into capital makes no change in the amount, it simply announces a new condition.

What condition? — That the advantage was not made by the capital in a single instant, but in that single period which the notation of the time expresses.

But what say you to times? — Neither does the multiplication of capital into a number of turns or times, change its amount; for turns are subdivisions of the unit of time, and have therefore a denominator; and this denominator is equal to the numerator, whatever be the subdivision; or the fractional form could not be equal to the unit, and the product would be false.

Of what use then is the multiplication into turns? — Capital is multiplied into turns when gain or loss is to be assigned to different parts of the time itself; so when one man's share is to be proportioned to that of others; because the magnitude of each subdivision of the time being the same, their absolute number, or their products, form a ground of comparison between the time of one share, and the time of another; between the product of one man's capital, and that of another.

What is the manner of it? — Every single share is, by the enunciation of it, a unit, and, multiplied into its time, remains a unit, though differently conditioned, having also made a certain portion of the gain and loss in question: this unit is justly represented by a ratio of equal terms, composed each of capital and time, as factors or as product.

How do you proceed? — Every unit, thus exhibited as a ratio, may, collectively, be formed into a single ratio by addition, and their sum total will be a unit still; namely, of larger value, having made the entire gain, or sustained the entire loss.

Can ratios be proportioned by addition? — *The addition of equal ratios makes no change in their proportional value*, for the terms are increased in exactly the same proportion in which they were before; thus, $\frac{1}{2}+\frac{5}{10}=\frac{6}{12}$, or $\frac{1}{2}$ still.

What may such a ratio be called? — A ratio formed of terms by addition, is not a conjoint, but a collective, ratio.

Lastly, how are the respective apportionments made? — The denominator of such a collective ratio remaining the same, if all the terms be removed from the numerator except what may belong to a single share, the fraction remaining will be the ratio of gain or loss assignable to that single share.

Why? — Because the unit is made up of all its parts, and every part has been removed, save one; this, multiplied into the entire gain or loss, will produce the proportional estimate sought, and assignable to the share represented by the numerator.

Now, can you exemplify this? — Suppose three shares, one of \$200, employed for 2 years; another of \$100, employed for 1 year; a third of \$50, employed for 6 months; and their profits to be 6 per cent, per annum; at this rate the gain assignable to \$50, will be \$1½; to the \$100 dollar share, it will be \$6; to the largest share, \$24; this is manifest.

How does it appear from the ratio? — The collective ratio will be, $\frac{\overline{50\times\frac{1}{2}}+\overline{100\times1}+\overline{200\times2}}{25+\ 100\ +400}$ of \$31½, entire gain; $=\frac{525}{525}$ of \$31½ $=\$31\frac{1}{2}\times1$. The ratio of the least share will be $\frac{50\times\frac{1}{2}}{525}$ of \$31½; $=\frac{25}{525}\times31\frac{1}{2}=21:31\frac{1}{2}=\$1\frac{1}{2}$; and so of the rest.

What is the denominator of these ratios? — The denominator of both ratios is the sum of the products of every share multiplied into its time; as may be seen by comparing the numerator of the first with its denominator, both being equal.

The author fears the patience of his readers may be exhausted before they get to the end of this demonstration; he has labored, to the utmost, to abridge it; but in no way more concise has he been able to make the subject clear to his own mind; and he shall think himself fortunate if he has made it clear to his readers; for, as to the explanations given of the case of double inequality and its solution, in the few books that treat of it, with which he is conversant, they afford but a curious example of the possibility of framing hypotheses which shall be true in theory, and false in fact. The maxims that follow, proceed, of course, on the supposition, that such are the terms on which a partnership has been entered into; as to the two last of them, perhaps a rare case now among a few individuals; but where profits are divided periodically in large institutions, it is presumed, that the principles expressed in these maxims are substantially acted upon.

Can you now recite the maxims?

MAXIMS IN PARTNERSHIP.

Equal shares and times entitle to equal distribution.

Unequal shares or times participate in the ratio of the sum of the shares or times to each share or time.

Shares and times both unequal participate in the ratio of the sum of the products of every share multiplied into its time, to the separate product of each share and time.

Proof. Equality of the sum of the portions found with the sum to be apportioned.

APPLICATION.

1. A contributes, toward a common stock, \$130; B, \$220; C, \$375; taking account at the end of the year, they find themselves losers of \$50; what must each man bear of the loss?

\$130 A's share.
220 B's
375 C's

a e \$725 stock.

A's loss is the ratio, $\frac{130}{725}$ of \$50, entire loss; $=\frac{65}{7\cdot25}=1\cdot45:13\cdot00$ [$=\$8\cdot965\frac{75}{145}$
B's, $\frac{220}{725}$ of \$50, $=\frac{110}{7\cdot25}=1\cdot45:22\cdot0=15\cdot172\frac{60}{145}$
C's, $\frac{375}{725}$ of \$50, $=\frac{187\cdot5}{7\cdot25}=2\cdot9:75=25\cdot862\frac{10}{145}$ Answer.

\$50·000 proof.

The fraction of C's portion is $\frac{2}{29}$ of the last decimal, equal to $\frac{10}{145}$ of the same. It might be supposed, that any last remaining apportionment could be found by subtraction; but no one portion can safely be subtracted till every other has been proved correct; and this proof can only arise from all the portions, obtained by a similar but independent operation, forming together a sum equal to that which is to be apportioned.

2. Three graziers hired land in common for \$60½. A put in 5 sheep during 4½ months; B, 8 during 5 months; C, 9 for 6½ months; what would be an equitable apportionment of the rent?

4½*m.* ×5=22·5 A's proportion.
5 ×8=40 B's
6½ ×9=58·5 C's

a e=121 sum total of the proportions.

A's rent is the ratio, $\frac{22\cdot5}{121}$ of \$60·5, entire rent; $=121:1341\cdot25$ [$=\$11\cdot25$
B's, $\frac{40}{121}$ of \$60·5, $=121:242=20\cdot$
C's, $\frac{58\cdot5}{121}$ of \$60·5, $=121:3539\cdot25=29\cdot25$ Answer.

\$60·5 proof.

Examples to be wrought, proved, and recited.

1. Two shares, respectively of $1200 and $359, have gained, in trade, $430·739; what is the proportional assignment of profit?

2. Three persons join their respective capitals of $750, $578, $219, and gain, in a twelvemonth, $521·51; what is the gain of each?

3. Two persons unite their stocks and their business, and gain $2179; of which A, for certain causes, is to have $63 per cent; B, the remaining $37 per cent; what was the share of each?

4. A and B make, by joint trading, $1300·795, on which B is to have $9 per cent more than an equal share, for special skill and attendance; what is the gain of each?

5. Three partners lose in trade $1000; the share of stock brought by A was $1750; by B, $980; by C, $2001; what is the diminution of capital to each?

6. Three partners gain, in trade, $1100·979; A had contributed to the common stock $\frac{7}{15}$ of the whole; B, $\frac{2}{5}$; C, the remainder; what is the gain of each?

7. A meadow is to be fed off by cattle belonging to three persons; A has put in 7 head of oxen; B, 9; C, 4 yoke; what must each pay?

8. Two persons buy a section of land between them; one pays $238; the other, all that remains from 1\frac{3}{4}$ per acre; what is the portion of land assignable to each?

9. A unites his capital of $3000 to that of B, which is $2108; after trading during 4 months, they take C as a third partner, with $1000; and find at the end of the twelvemonth, that their gain amounts to $1900·73; what is the share of each?

10. Three persons having engaged in trade, B and C uniting themselves successively to A; namely, B, at the end of 2 months, with $817; C, of 7 months, with $900; A's capital amounting to $750; find, at the end of the year, that they have incurred a loss of $315·14$\frac{3}{8}$; what is the loss to each?

11. A farmer rents a meadow during the season for $36, and drives into it 6 head of cattle; at the end of a fortnight he admits 7, and at the end of a month, 4, belonging to others; should these continue during the season, and the rent be proportionally divided, what will each have to pay?

12. A, B, and C, unite their capitals in trade; A, $2000; B, $966; C, $789; at the end of 3 months, C makes his share equal to B's, and at the end of 5 months, B makes his

share equal to A's; out of $2194·553, profit made during the year, what is each entitled to?

13. Three persons having contributed, one $655, another, $816, the third, $970, to a certain trade, what proportion of gain or loss will accrue to each?

The learner will now return to Duodecimals, that he may solve the proportional questions which he was there directed to postpone.

Imposts and Taxes.

What are the means of raising a public revenue? — Imposts and taxes form the chief part of the public revenue.

How are these distinguished? — Imposts are dues of custom and excise; taxes, commonly so called, are inland dues, levied either on person, or on property not taxed as an article of manufacture; this kind of tax passing by a different name.

Why are taxes levied? — Taxes, of whatever description they may be, are levied to defray the necessary expenses of government; and to discharge the interest, or capital, of money lent to a nation, in order to meet those expenses, when the nation itself was in want of the means.

What are the advantages derived from government? — Without government, we should be savages, possessing no security of life from public protection; no property in any thing which we did not hold in our hand, or watch with our eye; no knowledge beyond that of the first beast of the forest.

Has not the savage the advantage over the civilized man in respect of liberty? — There is no liberty in savage life, except the liberty to be wretched.

What seems to be the best mode of levying taxes? — There seems no better mode of levying taxes than by customs, or duties on goods imported from abroad; because the consumption of foreign goods is voluntary; and the tax is paid by the consumer, in a manner, unknowingly.

But is it not right that men should knowingly contribute to the support of government? — What is right, and what is practicable, are often, through ignorance and wickedness, very different things; there are men who purchase highly-taxed articles without complaining, who yet think the taxgatherer's requisitions a robbery, and an infringement on liberty.

In what does excise differ from customs? — Excise dues are an inland tax, levied on home manufactures; more known and felt, therefore more disliked, than duties levied only at seaports, and received from a few merchants.

If excise duties be odious, why are they anywhere levied? — When the custom-house revenue is insufficient, other taxes must be imposed; a poll tax has not been less odious than an excise; and a property tax, though falling much more equally, is incomparably more inquisitorial.

What is the best course for the public? — In all these matters, a wise government will regard the usages and even the prejudices, of a country; and will render taxes as light as justice and sound policy admit of.

Is high taxation a proof of general distress? — High taxation itself is no proof of distress; on the contrary, where large taxes are easily collected, this is a certain sign of prosperity; and in some countries, taxes are comparatively nothing, because the people have no means of paying taxes.

What is a tariff? — Tariff is a Spanish term, probably, for a table of custom-house dues.

What means *ad valorem*? — *Valorem* is from a low Latin word, used to signify value; and the phrase *ad valorem*, in custom-house use, denotes the bringing up of commodities to a value, fixed by law, as that on which the duties shall be estimated.

What charges does the ad valorem valuation include? — Under the tariff law of 1832, the ad valorem valuation includes all charges, except insurance.

Under what rule of arithmetic do all such matters fall? — Imposts and taxes are estimated by the rule of single proportion direct.

When a rate per cent is sought, what is *a r*? — When a rate per cent is sought, the terms of the question show that 100 is the article requiring estimate.

What are the maxims?

MAXIMS OF ASSESSMENT.

Tax rate per hundred is, as the whole property estimated to the property tax entire, the estimate given.

Proof. The product of the article estimated, multiplied into the rate of unity, is equal to the whole property tax.

Duty per cent is the product of the customable amount multiplied into the rate of unity; and the customable amount is made up according to the rules under which it may fall.

APPLICATION.

1. A town is assessed in the sum of $3086·09; the male inhabitants, from 16 years old and upward, are 728, and the property is estimated at $67069; what is the tax per cent and per dollar, the poll tax being $1 per head?

$3086·09 whole assessment.
728·00 poll tax.

$2358·09 property tax.

The tax rate is the ratio,

$\frac{100}{67069}$ of $2358·09, p. t. entire; =67069 : 235809=$3·5159 rate.

Answer, $3·516 per cent; $·03516 per dollar.

67069×·03516=$2358·146 proof.

The proof is in excess, because the fraction of a mill in the quotient is taken as a mill.

2. What is the amount of duty on a box of window glass, containing 50 plates, 8*i*. by 10*i*. each, at $3 per hundred square feet?

8′×10′=6′ 8″
50

27⌉*f*. 9′ 4″ total sq. meas.

27⌉*f*. 9′ 4″=4000″.
100⌉*f*.=14400″.

$\left\{ \begin{array}{l} \frac{3}{14400}=·000208 \\ \frac{·8\dot{3}}{4000}=·000208 \text{ proof.} \end{array} \right.$

The duty sought is the ratio,

$\frac{27⌉f.\ 9'\ 4''}{100⌉f.}$ of $3, rate;

	27⌉*f*.
6′=½⌉*f*.	·03 rate of unity [(duty.)
	·81
3′=½ of ½	·015
4″=⅑ of ¼	·0075
	·0008$\dot{3}$
	$·8333$\dot{3}$ ans.

3. What is the amount of a 50 per cent duty ad valorem, on 87 square yards of woollen cloth, invoiced, at 17*s*. the

yard; charges, insurance excepted, amounting to 8*d.* the yard, and the pound sterling being valued at \$4·80?

Cost and charges per yard, 17*s.* 8*d.*

{ 6*d.* = ½*s.* | 87*yd.*
{ 2*d.* = ⅓ | 17*s.*

—

609
87

—

1479
43 6
14 6

—

2,0 : 153,7 0 = £76 17*s.* cost and charges entire.

{ 2 1 2
{ 1 8444 / 87 = 212 proof of cost and [charges.

£76·85 cost and charges.
·5 rate of unity (duty).

—

38·425
12 × ·4 = \$4·8 = £1.

—

461·100
·4

—

\$184·44 answer.

Examples to be wrought, proved, and recited.

1. What is the property tax per cent on \$228105·173, assessed at \$5000?

2. On property valued at \$96017, if the tax rate be \$1·11¼ per cent, what is the tax entire?

3. Where the taxable population is 1266, the assessment, \$3009·161, the property valued at \$80001, and the poll tax, \$·83, what is the tax rate per cent?

4. Where the taxable population is 10011, the poll tax, \$·62½, the property valued at \$318016·73, and the rate per cent, \$·991, what is the assessment entire?

5. What is the amount of duty on 3 bales of wool, averaging each 2 cwt. English; cost, \$·10 per ℔; charges, \$2·50 per bale; rate, \$40 per cent ad valorem?

6. What is the amount of duty on 5 pieces of Nankin, each containing 27 *yd.*, at \$·83 cost price per yard, and \$·05 charges; rate, \$20 per cent ad valorem?

7. What is the amount of duty on 63℔ of silver wire, invoiced at £3 14*s*. 4*d*. the pound weight, charges, 5*s*. 3*d*. the ℔; rate, £5 per cent ad valorem?

8. What is the amount of duty on 17 fowling-pieces, cost, 15 guineas each, charges, 4*s*. each; rate, £30 per cent ad valorem?

9. What is the amount of duty on 219 *yd*. of sail duck, at 2*s*. 7*d*. the yard, charges, 3*d*. the yard; rate, 15 per cent ad valorem?

10. What is the amount of duty on Q 2 3℔ of salt; rate, $·10 on every 56℔?

11. What is the amount of duty on 25 *yd*. of Brussels carpeting, at $·63 the square yard?

12. What is the amount of duty on 67℔ of sewing silk; cost, 40*s*. the ℔; whole charges, £1 18*s*.; rate, £40 per cent ad valorem?

13. What is the amount of duty on millinery; cost, £78 9*s*. 2½*d*., charges, £4; rate, £25 per cent ad valorem?

The rates of duty above are from the tariff law of 1832.

Stocks.

What is the denomination commonly given to property in the public debt? — As the property of individuals is called their funds; so claims against government, being in the nature of property, are called public funds; and the claimants, fundholders.

Have they not another appellation? — Fundholders are also called stockholders, because they have contributed to the capital stock which, at one time or another, has been loaned to government.

Is the same term made use of in any private establishments? — Shares, in moneyed, manufacturing, and other institutions, are also called stock, because the capital advanced is the stock of the establishment, on which its operations must be founded.

On what does the security of private stock rest? — The security of private stock depends, to a considerable extent, on the credit of the institutions to which it appertains; but, fundamentally, on the capital possessed, and the skill with which it is employed.

What then is the security of government stock? — The security of government stock is one altogether of credit; whatever pledges there may be in the nature of a legal security, the holders of public stock rely entirely on the wisdom and

integrity of rulers and legislators; and on the justice of a nation, for whose aid they, or their ancestors, have contributed their private property.

What influence have its debts upon a nation? — An excessive debt is a great national evil, in the weight of taxes necessarily imposed to defray the interest, to discharge the principal, and to meet the expenses of management.

Is any good derived from public debt? — A national debt restrains the disposition of mankind to war, and affords the securest investment of property in behalf of the widow and orphan, the aged and infirm, who are often bereaved of their support, by confiding their all to private establishments, and the mercy of individuals.

How may an accumulation of public debt be prevented or corrected? — National debts are prevented chiefly by the avoiding of war, and can be remedied only by strict economy in every department of state.

What is the rate of interest paid on government stock? — The interest paid on government stock is commonly the lowest, for the capital lent, of any paid in a country; because the security of it has always been confided in, as the best which a country's honor can afford.

What are the terms used in stock transactions? — Shares, par value, premium, discount, and dividends, are the principal terms of the stock exchange.

What is meant by par? — *Par* is a Latin word, signifying equal; and to be at par, imports an equality of real and nominal value.

Do these often differ? — Stocks are seldom at a valuation equal to the amount which any portion of stock nominally bears; but are above or below par.

Whence their variableness? — The value of public stock depends ultimately on the stability of the government; but the constantly operating causes of variation in price are the national expenditure, the state of commerce, the increase or diminution of the precious metals.

In what manner do these circumstances operate? — It is an acknowledged principle in human transactions, that the estimated value of an article increases with its scarcity and the greatness of the demand; diminishes consequently with plenty and a small demand.

What is the inference? — Therefore when money is little in demand for public expenditure and commerce, the demand is great for stock, that capital may obtain interest; stock then

When otherwise? — When however money is greatly in demand by government, new stock is created; when commerce is very flourishing, many are desirous of parting with stock, to engage in it; the effect of either circumstance is an increase of stock for sale, and a diminution of price.

Is stock divided into fixed portions? — Stock may or may not be divided into shares; but government stocks, it is believed, never are; and may therefore be purchased to a very small amount.

What does the word *par* specially distinguish? — *Par* is always used of the nominal amount of stock.

What is the interest on stock called? — The name given to interest on stock is, its dividends, which are usually paid every half year.

What are premium and discount? — Premium is the excess of price above par; discount, the depreciation below it.

How is brokerage charged in stock transactions? — Brokerage is charged at so much per cent on the par amount of stock purchased or sold; added to the charge on a purchaser, subtracted from the proceeds to a seller; but not charged both ways by the same broker.

What rule governs stock transactions? — *All estimates of stock are determined and proved by the maxim and proof in proportion direct.*

N. B. In calculations of stock, great attention must be paid to the right distinguishing of the terms of the statement.

APPLICATION.

1. What is the price of 5½ shares of U. S. Bank stock, at $111?

$111	current price.
5½	shares.
55·5	
555	
$610·5	answer.

2. How many shares of a joint stock, at \$9 per cent premium, can be purchased for \$700?

$e\ s$ is the ratio,

$\frac{700}{109}$ of \$100, par; $=109 : 70000=\$642{\cdot}2018$, price. $\Big\}$ Ans.
$6{\cdot}422018$, shares.

$a\ r=\$700$ current price.
$a\ e=\$109$ same.
$e\ g=\$100$ par.

$\left\{ \begin{array}{l} \frac{100}{109}=\cdot9174 \\ \frac{642{\cdot}2018}{700}=\cdot9174, \text{ proof.} \end{array} \right.$

The answer, *in shares*, is 6, and a considerable part of a 7th; for the premium being mentioned at a per centage, shows that the shares are \$100 each.

3. What is the price of 13 shares in a joint stock concern, at a discount of \$3 on \$50, the par value?

$\$50-3=\$47. \quad \$47\times13=\$611$, answer.

4. What amount of 4 per cent British stock, at £96 per cent, can be purchased for £378?

$a\ r=$£378 current price. $\frac{378}{96}$ of £100, par; $=12:37800$
$a\ e=$£ 96 same.
$e\ g=$£100 par. $8 : 3150$

£393 15*s*. ans.

$\left\{ \begin{array}{l} \frac{100}{96}=1{\cdot}041 \\ \frac{393{\cdot}75}{378}=1{\cdot}041, \text{ proof.} \end{array} \right.$

5. What will be the broker's bill on the purchase of \$600, U. S. 5 per cent stock, at \$107; brokerage, \$¼ per cent?

$e\ s$ is the ratio, $\frac{600}{100}$ of \$107, current price; $=\$642$
$1{\cdot}50$ brokerage [added.
$\$643{\cdot}50$ answer.

6. What will be the proceeds to the seller of the same, on the same terms?

The statement being the same, $\$642-1{\cdot}50=\$640{\cdot}50$, ans.

Examples to be wrought, proved, and recited.

1. What is the price of \$3360 U. S. 5 per cent stock, at \$20 per cent premium?

2. What is the price of 19 shares of U. S. Bank stock, at \$104?

3. What is the value in federal money of £5068 13*s*. 6*d*., British 3 per cent stock, at a discount of £18¾ per cent?

4. What amount of U. S. Bank stock, at \$103¼ per cent, can be purchased for \$1202·112; and what will be the broker's bill, at \$¼ per cent brokerage?

5. How many shares in a joint stock, at the par of \$50 per share, can be purchased for \$1609; and what will be the broker's bill, allowing $\$\frac{1}{4}$ per cent brokerage?

6. What is the price of 16 shares in a joint stock, at \$25 per share, when at a discount of $\$1\frac{1}{8}$?

7. What is the price of 11 shares in a joint stock at \$150 per share, a premium of $\$6\frac{1}{8}$ per cent, and a brokerage of $\$\frac{1}{4}$ per cent?

8. What amount of federal money will purchase £5000 worth of Bank of England stock, if the current price be £$199\frac{4}{7}$ per cent, and the pound sterling be estimated equal to \$4·80?

9. What amount of joint stock, at \$75 per share, and a premium of $\$2\frac{1}{12}$, can be purchased for \$565?

10. What is the price of 25 shares in a joint stock, at \$35 per share, but depreciated \$8 per cent?

Inverse.

What questions fall under the rule of inverted proportion? — Calculations of bodily labor and instrumental operation, to be increased with the diminution of time, or diminished with the increase of time, are the most frequent subjects of the rule of inverse proportion.

Can you generalize your statement? — Inverse proportion governs every case of a limitation, the selfsame, applying to both articles estimated and requiring estimate; for as one article is lessened, it will require a greater estimate than the other article; as one is augmented, it will require a smaller estimate.

Why should it be so? — The limitation being the same to both, or the effect to result from both being the same, it is manifest, that as one means of producing the effect, or of reaching the limitation, is reduced, some other means must be increased; the reduction therefore of an article must be attended with an increase of estimate in something equally effective; and the increase of an article with a diminution of the estimate; or the limitation will be exceeded.

Can you exemplify it? — A structure, calculated to be built in a certain time, must have more hands employed upon it if the time is to be diminished; a specific amount of interest may be earned, by increasing the time, though you lessen the principal.

But how do you apply these instances to the limitation? — The structure and the interest are the effects limited to one

period of time, or to a different period of time, according as their estimates in workmen and principal money are changed.

How are the articles to be distinguished? — The articles in proportion inverse are the terms whose increase or diminution is known, because expressed in the question; and necessarily to be known, since from them the unknown diminution or increase of estimate is to be found.

Does any distinct head of accounts fall under this rule? — Equation of payments, at simple interest, proceeds entirely on the principle of proportion inverse.

Do you recollect the maxim and proof?

MAXIM IN PROPORTION INVERSE.

The estimate sought in proportion inverse is the ratio $\frac{a\,e}{a\,r}$ (denominated alike), of the estimate given.

Proof. The inverse ratio of the articles is equal to the ratio of the estimates direct thereto.

APPLICATION.

How much cloth, $\frac{7}{8}$*yd.* wide, is required to line $4\frac{3}{4}$*yd.* of cloth, of 5*qr.* width?

$$a\,r = \tfrac{7}{8}yd.$$
$$e\,g = 4\tfrac{3}{4}yd.$$
$$a\,e = 5qr. = \tfrac{10}{8}yd.$$

e s is the inverse ratio,

$\dfrac{\frac{10}{8}}{\frac{7}{8}}$ of $4\frac{3}{4}$*yd.*, length given; $=\frac{10\times4\frac{3}{4}}{7}=7$: 47·5=6*yd.* $3\frac{1}{7}$*qr.* ans.
42
———
5·5
4*qr.*=1*yd.*
———
: 22·0
21
——
1

$\left\{\begin{array}{l}\frac{10}{7}\text{ inverse r. of art.}\\ \frac{27\frac{1}{7}qr.}{19qr.}\text{ d. r. of e.}=\frac{190}{133}=\frac{10}{7}\text{ proof.}\end{array}\right.$

How much cloth here signifies, what length of cloth; and the inversion of the proportion proceeds from the circumstance, that the wider the lining cloth is, the less is needed in length. With a tabular divisor, we have used long division, as the question required a reduction of remainders to smaller composites.

Examples to be wrought, proved, and recited.

1. If 29 workmen could erect a building in 3 *mo.* 5 *da.*, how many are requisite to complete the structure in one month ?

2. If 17 men can execute a work in 63 days, what is the largest number necessary for its accomplishment in one third of a year ?

3. A building has been erected by 15 men in 5 months ; how many are needed to erect one twice as large in $\frac{1}{7}$ of a year ?

4. By travelling during 8 hours of the day in summer, a person was able to perform a journey of 1228 miles in 36 days, including 5 sabbaths, on which he rested ; how long will he be in returning, during the winter season, travelling, at the same average of speed, only six hours of the day, beginning his journey on a Monday, and resting on the sabbaths ?

5. If 20 boarders drink 9 gallons of cider in a week, how long will the same quantity last for 11 boarders, at the same rate of consumption ?

6. A vat of beer, containing 19 *hhd.*, has 2 faucets, of different bores, near to the bottom of the vessels ; one empties the vat in 2 hours 16 minutes, the other, in 59 minutes ; if turned together, in what time would the two empty it ?

7. A ship's company, reduced to the allowance of $1\frac{1}{2}$ pint of water during every 24 hours, on a calculation that it will hold out for 3 weeks, deem it necessary, at the end of a week, that the allowance shall be still farther reduced, to make the water hold out as for 3 weeks from that time ; what was the allowance reduced to ?

8. A loaf of a certain price weighs ℔ 17 *oz.*, when flour is $\$6\frac{5}{8}$ per barrel ; how much will it weigh when flour is at $\$8\frac{1}{2}$ per barrel ?

9. What sum, at interest, will earn as much in 3 months, as \$719·21 gain in 9 months ?

10. A besieged city contains 17024 inhabitants, with flour sufficient to average 1℔ per diem to each, during 4 months ; what reduction of the allowance is necessary, if they contemplate a protraction of the siege during 3 weeks longer ?

11. What quantity of cloth $\frac{5}{8}$ *yd.* wide will line $17\frac{1}{4}$ *yd.* of cloth, yard wide ?

12. A regiment consisting of 1000 men is to be supplied with coats ; the average quantity of cloth in each coat is $2\frac{1}{2}$ *yd.*, of $1\frac{5}{8}$ *yd.* width ; if lined completely with cotton cloth $\frac{7}{8}$ *yd.* wide, how many yards of lining will be required ?

13. If 3 mowers, and 8 haymakers, can harvest the hay on a farm in $3\frac{1}{2}$ weeks, how many are needed to accomplish the work in a fortnight, observing the same proportion ?

14. If a board be $1\frac{1}{2}$ *ft.* wide, how long must it be to measure 13 feet square ?

15. What breadth is requisite, with 7 inches of length, to make a foot square ?

16. How many yards of matting, 1 *ft.* 10*i.* wide, will cover a floor, 17 *ft.* by 12 *ft.* 11*i.* ?

17. If a certain distance can be travelled over in 23 days, when the day is 11 hours long, what will be the number of days' journey, when there are 14 hours of daylight, using the same speed ?

Conjóint.

Do you recollect what conjóint proportion was defined to be ? — Conjóint proportion is the mode of determining an estimate, by the joining together of two or more ratios in one, when more than a single comparison is made.

What was said of conjóint proportion direct and inverse ? — Proportional operations uniting both direct and inverse ratios may perhaps be called mixed proportion.

What are the maxims ?

MAXIM IN CONJÓINT PROPORTION.

The estimate sought in conjóint proportion is the ratio $\frac{\text{ar}\times\text{ar}}{\text{ae}\times\text{ae}}$ (denominated alike), of the estimate given.

Proof. The conjoint terms are in equal proportion to their respective estimates given and found, constituting similar ratios.

MAXIM IN MIXED PROPORTION.

The estimate sought in mixed proportion is the ratio $\frac{\text{a r}\times\text{e}}{\text{a e}\times\text{r}}$ (denominated alike), of the estimate given.

Proof. The mixed ratio of the articles, as it appears in the operation, is equal to the ratio direct of the estimates given and found.

APPLICATION.

If a journey of 130 miles be accomplished in 3 days, when the days are twelve hours long, in how many days of 10 hours may 360 miles be travelled at the same speed?

$a\,e = 130ml.$; direct.
$e\,g = 3da.$
$a\,e = 12h.$; inverse.
$a\,r = 10h.$; inverse.
$a\,r = 360ml.$; direct.

The days sought are the mixed ratio,

$$\frac{360\times12}{130\times10}\text{ of }3da.; = \frac{3\cdot6\times12\times3}{13} = \frac{10\cdot8\times12}{13} = 13:129\cdot6$$
$[= 9\cdot96923da.$ ans.

$$\left\{\begin{array}{l}\frac{3}{1300} = \cdot0023\\ \frac{10}{4220} = \cdot0023 \text{ proof.}\end{array}\right.$$

Examples to be wrought, proved, and recited.

1. If 8 men can trench 27 *yd.* in 3 days, how many are needed to trench 150 *yd.* in 9 days?

2. If the expenditure of a family of 7 persons, during $7\frac{1}{2}$ months, be \$696, how much will maintain a family of 9 persons, at the same rate, during $17\frac{1}{3}$ months?

3. If 14 workmen earn \$150 in 17 days, what will be the wages, at the same rate, of 19 workmen, during $15\frac{1}{2}$ days?

4. If 60 bushels of grain be sufficient for 7 horses during 28 days, how long will 108 bushels serve 13 horses?

5. If 1500℔ of beef serve 170 men during $7\frac{1}{2}$ days, how much will be required for 1000 men, during 2 months?

6. If a barrel of cider, of $31\frac{1}{2}$ *ga.*, be sufficient for a family of 10 persons during 3 weeks, how much may suffice a family of 5 persons from the autumnal to the vernal equinox?

7. If 90 men in 12 days of 10 hours each, can trench 200 yards in length, 3 *yd.* in width, and 2 *yd.* in depth; in how many days of 8 hours long will 133 men trench 573 *yd.* in length, 4 *yd.* in width, and 3 *yd.* in depth?

8. If 27 men can reap 165 acres in $9\frac{1}{2}$ days, how many acres may 13 men reap in 19 days?

9. If a garrison of 200 men be supported, during $\frac{3}{4}$ of a year, at an expense of \$4015, how long may 63 men be supported for \$1000?

10. If a pound of bread cost \$$\cdot06\frac{1}{4}$, when flour is \$11 a barrel, how much may be purchased for \$$\cdot23$, when flour is \$$9\frac{1}{3}$ a barrel?

Interest.

Distinguished from usury.

On what conditions do men commonly lend money?—The usual condition of loans is the payment of interest.

Then what is interest?—*Interest is the legal hire of money lent.*

How is this distinguished from income?—Income is a man's yearly revenue from whatever source it may accrue.

Why did you say legal hire?—Usury is unlawful hire of money lent; or, where there is no law on the subject, excessive hire.

When is the hire of money excessive?—The hire of money is excessive, whenever it is unreasonable for the country and the occasion.

What has occasioned the regulation of interest by law?—A wish to protect individuals from the consequences of their own ignorance, and from seductions to vice, are the motives that should operate with government to interpose its authority in the regulation of interest.

Are these motives fortified by any peculiar considerations?—Men acquainted with history know that whole nations nearly, and that for ages, have been rendered miserable by usury.

Yet ought we not to leave pecuniary, like other matters to self-regulation?—If money were like other matters, its hire would best be left to private control.

In what does it differ?—Money is the same with other commodities, when treated as such in the way of barter for goods, as in common transactions; or in whatever manner employed, in good faith, for the uses of commerce; being however a convenient and universal medium of exchange, it is the great and ready instrument of vice and folly.

What was anciently thought of interest?—All hire of money was, in very early ages, considered equally just or the contrary, and was denominated usury; but those were times in which little money was afloat, its value consequently enormous; and when a failure in making it good was fatal to the happiness and liberty of the borrower.

What is the character of modern interest?—The interest limited by law under Christian governments is moderate, and highly promotive of the general welfare, by affording facilities to fair trade and safe means of investing capital.

Has any mischief resulted from public interference in these *matters?*—It is difficult to imagine any mischief to have

arisen from the authoritative regulation of interest; for it appears from history to have followed the course of trade, not to have forced it.

Entirely as the author approves of the regulation of interest by public authority, he considers any penalty on usury beyond the forfeiture of all interest, received or due, as exceeding the measure of the offence, and inducing to acts of baseness.

Diversity of interest.

What are the diversities of interest? — Interest is simple or compound.

What is the first? — *Simple interest is the hire accruing on the principal only, or money actually lent.*

What is the second kind? — *Compound interest is interest calculated on the principal sum due, and on the continually accruing interest, unpaid, and added to the principal.*

What is the morality of it? — Compound interest, though not legal, is not less just and fair than simple interest; for interest unpaid at the time it is due, becomes money lent; or rather, in most cases, money forcibly and unjustly detained; a circumstance that can entitle to no favor.

What is discount? — Discount proper is an allowance made on the payment of money before it is due; less than interest, but proportioned, after the same manner, to the use of money for a time. The term discount is also used for interest itself, taken in advance before it becomes due, and for interest allowed in the place of discount proper, on payment anticipated.

Terms, &c.

What are the terms employed in this rule? — The terms employed in money dealing are principal, or the sum lent; interest, or the hire; rate, the hire per cent, per annum; and amount, or sum total of principal and interest.

What is the signification of the words used with rate, or implied in it? — *Per cent,* is a Latin phrase abbreviated; meaning, by the hundred; *per annum* is a Latin phrase, meaning, by the year.

Has rate any peculiar meaning in transactions of interest? — *The rate of interest is the estimated hire of a hundred, during a year.*

By what rule are these questions determined? — All questions of interest and discount are determined by the rule of conjòint proportion; two comparisons being made, of principal with principal, and of time with time.

Why should time be an element? — The longer the time, the greater is the advantage ordinarily accruing from money in use.

How is time computed in respect of interest?

BANKING TIME.

3 days over time are days of forbearance.
30 days are 1 month.
90 days are 1 quarter.
360 days are 1 banking year.

What is meant by days of forbearance? — Days of forbearance are days over the stipulated time of payment, during which a promissory note is not enforced by law, though chargeable with interest for the delay.

Have times any particular commencement in such transactions? — Banking time is reckoned from any day, to the 30th, 60th, or 90th following, without reference to the calendar.

SIMPLE INTEREST.

What are the questions arising under the head of simple interest? — Under simple interest, the estimate sought is chiefly interest itself; occasionally, principal, rate, or time.

Whence is interest sought? — Interest is sought from principal, rate, and time; for it is proportioned to all three.

How is the ratio formed? — The ratio of interest is, as 100 to the product of principal and time, the rate being the estimate given; for the rate is interest of the hundred during a year.

But where is the time in *a e*? — When the article estimated is the interest hundred, the time estimated is usually 1 year; it is therefore needless to conjoin the time unless in cases of reduction to some lower denomination, made necessary by the article requiring estimate.

Whence is the principal sought? — The principal is sought from amount, rate, and time; from the amount, because this consists of principal and interest added together; but to subtract the interest, we must know the rate and the time.

What will be *e g*? — Principal being sought, 100 is the principal given; for that is a principal sum which gives the rate.

When sought from income, what is *a e*? — When principal is sought from income, the article estimated is the rate; for the rate is the income of a hundred.

Whence is the rate ascertained? — If the rate be unknown, it is sought by subtracting the principal from the amount, the

difference is the interest; this, proportioned to a hundred and a year, gives the rate.

What will then be *e g*? — The rate being interest, the interest arising on the principal sum for the whole time will be the estimate given.

Whence the time? — The interest of a hundred for the whole time having been found, if divided by the rate, we have the time; for interest is the rate multiplied into the time.

What will be *e g*? — When the time is sought, one year is, ultimately, the time given.

Six per cent interest.

What rate of interest is most usual in the U· S·? — Interest most usual in the U· S· is 6 per cent.

At this rate, what will every hundred gain in two months? — At six per cent, per annum, every hundred dollars gains one dollar in every two months; for 2 months are $\frac{1}{6}$ of a year, and therefore give $\frac{1}{6}$ part of \$6, the rate.

What will be the interest for a day? — In 12 banking months there are 360 days; the interest, consequently, of \$100, for one day, will be $\frac{1}{360}$ of \$6, or $\frac{6}{360}$; this, reduced to its lowest terms, is $\frac{1}{60}$ of a dollar.

Are these results employed in the finding of interest? — At \$6 per cent, annual interest, a principal sum, divided by **100**, then multiplied into half the number of even months, and as many sixtieths as there are days over, will produce the whole interest for months and days.

How is this demonstrated? — The denominator of the interest ratio is **100**, therefore the principal must be divided by **100**; the quotient is a number of hundreds, or parts of a single hundred, or both; every single hundred, and in proportion for its parts, gains \$1 every 2 months, and \$$\frac{1}{60}$ for every day over; the quotient therefore is multiplied into half the number of even months, and as many sixtieths as there are days over, including any odd month.

What, in short, is the principle of this method? — The numbers assumed, as of months and days, are the product of rate and time.

Can you demonstrate it by example? — Let \$200 be the principal; 5 months the time; the rate, \$6 per cent; then, because the time is months, or $\frac{5}{12}$ of a year, the ratio has a complex nu-

$\frac{200\times\frac{5}{12}}{100}$ of \$6, $=2\times\frac{5}{2}=\$5$;

Or,

$\frac{200\times\frac{5}{12}}{100}$ of \$6, rate; $=2\times\overline{\frac{4}{12}+\frac{30}{360}}\times6=2\times\overline{\frac{4}{2}+\frac{30}{60}}=2\times2\frac{30}{60}=2\times2\frac{1}{2}=\$5.$

merator; but as the rate measures the complicator, we might obtain the estimate sought, which is $5, instantly; but successive reductions show, that the factors may become the hundredth part of the principal, half the number of even months, and $\frac{1}{60}$ of all the days over.

Does this mode serve for other rates of interest? — Interest at other rates may be found by this method, if we add or subtract parts of the estimate at $6 per cent, proportional to the difference of interest.

Can you exemplify this? — Interest at $7 per cent is obtained by adding one sixth part of itself to the estimate found at $6 per cent; for the difference of the factors is $\frac{1}{6}$ of 6; the difference therefore of their products will be in the same proportion.

How is interest found for years and parts? — Interest for years is first found, whatever be the mode; the interest of parts is then added to it.

Sterling money.

How is interest found on sterling money? — Interest on sterling money is found, by taking aliquot parts of the rate in a hundred pounds, multiplying them into the principal, and their product into years and aliquot parts of a year.

Must not the principal be divided by 100? — By multiplication into aliquot parts of the hundred, the principal is divided by 100; for the aliquot parts are reduced ratios, equivalent to others having 100 for their denominator.

Can you give an example? — Five per cent are $\frac{5}{100}$, equivalent to $\frac{1}{20}$.

What are the maxims?

MAXIMS IN SIMPLE INTEREST.

General. Interest sought is the product of the principal and a hundredth part of the rate, multiplied into years and fractions of a year, decimally or by aliquot parts.

Six per cent. Interest for months and days, at $6 per cent, is the product of the hundredth part of the principal, multiplied into a number equal to half the number of even months, and to as many sixtieths as there are days over.

Proof. The conjoint terms are in equal proportion to their respective estimates, given and found; constituting similar ratios.

Principal sought from amount, rate, and time, is, as the *amount of* a hundred for the rate and time, to a hundred, the *principal given.*

Proof. The principal found, added to its interest for the rate and time, is equal to the amount given.

Principal sought from income is, as the rate to a hundred, the principal given.

Proof, as in proportion direct.

Rate and time, or interest of a hundred for the whole time, are, conjointly, as the principal estimated to the interest given by it; severally, the *rate*, as the time, and the *time*, as the rate, to the entire interest of a hundred.

Proof, in either case, from the rate and time to the interest entire.

APPLICATION.

1. What is the interest on \$789·156, during 5*yr.* 7*m.* 17*da.*, at \$6 per cent, per annum?

By the general maxim.

\$789·156 principal.
·06 rate of unity.

47·34936 int. for a year.
5*yr.*

236·74680 — 6*m.* = $\frac{1}{2}$*yr.*
23·67468 — 40*da.* = $\frac{1}{9}$
5·26104 — 6*da.* = $\frac{1}{60}$
·789156
·131526 — 1*da.* = $\frac{1}{6}$ of $\frac{1}{60}$

\$266·603202 answer.

By the 6 per cent maxim:

\$789·156
·3 = 5*yr.* × ·06

\$236·7468 int. for 5*yr.*

$\frac{1}{2}$ of 6*m.* = 3
1*m.* 17*da.* = 47 days.

\$7·89156 hundredth of principal.
$3\frac{47}{60}$

5524092
3156624

6,0 : 370·90332

6.181722
23·67468

29·856402 int. for mo. and days.
236·7468 int. for 5*yr.*

\$266·603202 answer.

e s is the conjoint ratio,

$\frac{789{\cdot}156 \times 5\frac{227}{360}yr.}{100 \times 1yr.}$ of \$6, rate.

$5\frac{227}{360} = 5{\cdot}63\frac{1}{180}yr.$

$\frac{6}{100} = {\cdot}06$

$\frac{266{\cdot}603202}{4443{\cdot}3867} = {\cdot}06$ proof.

The proof in conjoint proportion peculiarly needs a deliberate consideration of the statement; the process above may be thought exceedingly operose, but it is the direct proof, and, from the nature of interest statements, every other mode seems to involve the principal operation. The common fraction of the numerator is reduced to a decimal, for more convenient multiplication; and the proof is exact. The learner will be able to judge, by a comparison of the principal work under each maxim, which is the simplest and most intelligible in operation, as well as expression. The 6 per

cent maxim is indeed often most erroneously enunciated; and it is surprising, if frequent miscalculations are not the consequence. The author submits to his readers, whether it be not easier to recollect the constantly recurring multipliers, $·06 for the rate of unity, $\frac{1}{4}yr.$ for 3 months, $\frac{1}{2}yr.$ for 6 months, ·75*yr.* for 9 months, $\frac{1}{120}yr.$ for 3 days, and to operate with them, than to operate with the perplexing fractions often given by a sixtieth part of the days.

2. Principal, £525 17*s.* 2¼*d.*; rate, £3½ per cent, per annum; time, 9*m.* 3*da.*; what is the interest?

$\left\{\begin{array}{l} £2\frac{1}{2} = \frac{1}{40} \text{ of } 100 \\ 1 = \frac{1}{100} \end{array}\right.$	£525	17*s.*	2¼*d.*	principal.
	13	2	11	
	5	5	2	
6*m.* = ½*yr.*	18	8	1	int. for a year.
3*m.* = ½ of ½	9	4	0½	
3*da.* = $\frac{1}{30}$ of ¼	4	12	0¼	
		3	0¾	
	£13	19	1½	answer.

17*s.* 2¼*d.* = £·859$\dot{3}$

e s is the conjoint ratio,

$$\frac{525·859\dot{3} \times \overline{\tfrac{3}{4}yr. + \tfrac{1}{120}yr.}}{100 \times 1yr.} \text{ of } £3·5$$

$$\frac{3·5}{100} = ·035$$

$$3da. = \tfrac{1}{120}yr. \;\big|\; 525·859\dot{3}$$
$$\tfrac{3}{4}yr.$$

4 : 1577·5780

394·3945
4·38216$\dot{1}$

3,9,8·77666$\dot{1}$: 13·957183 = ·035 proof.
119

20
20

We have in this example a specimen of the difficulties which small aliquot parts of the hundred a nd the pound sterling, indeed

of any composite, involves us in, when multiplied into denominations; since all such factors with ciphers, $\frac{1}{10}$ only excepted, require as much side work as any other factor not a tabular number. The facility with which the same work is performed by a reduction of the lower denominations to decimals of the higher will be seen in the next example. When aliquot parts, such as we have defined them, can be obtained, the denominations are often best preserved; but then only. The formidable array of figures in the proof vanishes by the use of contracted decimal division.

3. Principal, £963 13*s.*; rate, £4½ per cent, per annum; time 1*yr.* 7*mo.* 5*da.*; what is the interest?

$$\begin{array}{r|l}
 & £963\cdot65 \text{ principal.} \\
 & \quad \cdot05\times\cdot9=\cdot045 \text{ rate of unity.} \\
\hline
 & 48\cdot1825 \\
 & \cdot9 \\
\hline
6m.=\frac{1}{2}yr. & 43\cdot36425 \text{ int. for a year.} \\
1m.=\frac{1}{6}\text{ of }\frac{1}{2} & 21\cdot682125 \\
5da.=\frac{1}{6}\text{ of }\frac{1}{12} & 3\cdot6136875 \\
 & \cdot60228125 \\
\hline
 & £69\cdot26234375 \\
\hline
 & 5s.\ 2\tfrac{3}{4}d. \\
\hline
 & £69\ 5s.\ 2\tfrac{3}{4}d. \text{ answer.}
\end{array}$$

The interest sought is the conjoint ratio,

$$\frac{963\cdot65\times1\frac{215}{360}yr.}{100\times1yr.} \text{ of } £4\cdot5, \text{ rate.}$$

$$\begin{cases} \frac{4\cdot5}{100}=\cdot045 \\ \frac{69\cdot26234375}{1539\cdot163319}=\cdot045 \text{ proof.} \end{cases}$$

No difficulty occurs in the extemporaneous manner of reducing decimals of the pound sterling to composites; £·25 we instantly recognise to be equal to 5*s.*; the remaining ·0123 to be equal to 11 farthings; because $\frac{3}{10}$ are not equal to $\frac{12}{24}$.

4. What principal sum will amount to $1000, in 21 years, at $6 per cent, simple interest?

21*yr.*
6 rate.

126
100

$226 amount of $100 in 21*yr.*

The principal sought is the ratio, $\frac{1000}{226}$ of $100, principal given; = 2·26 : 1000=$442·4778, ans.

$442·4778 principal found.
·06 rate of unity.

26·548668
7*yr.*×3=21*yr.*

185·840676
3

557·522028 int. for 21*yr.*
442·4778 add principal.

$999·9998 proof.

The inquiry, what principal? is of frequent occurrence, and great importance, in human affairs; comprehending, among other things, all investments, for a time fixed, in behalf of offspring. Such questions are usually determined by calculations at compound interest, and will again be considered when we treat of that rule; but compound interest is not always realized, and it may be well to compare the results in both manners.

5. At what rate of simple interest will $442·4778 amount to $1000, in 21 years?

$1000 amount.
442·4778 principal.

$557·5222 int. for 21*yr.*

The interest of $100 for 21 years, is the ratio $\frac{100}{442\cdot4778}$ of $557·5222, int. given; 442·4778 : 55752·22=$126. The rate is, as $\frac{1yr.}{21yr.}$ of $126, int. of 100; =$6 ans. Proof as above.

6. In what time will \$442·4778 amount to \$1000, if placed at interest of \$6 per cent, per annum; reckoning simple interest only?

Interest of 100, found as for the rate. The time is, as $\frac{\$126}{\$6}$ of 1*yr.*, [time given; =21*yr.* answer. Proof as above.

Examples to be wrought, proved, and recited.

1. What is the interest of \$709·16$\frac{2}{8}$, for 5*mo.* 1*da.*, at 6 per cent, per annum?
2. What is the interest of \$1900·538, for 7*mo.* 19*da.*, at 6 per cent, per annum?
3. What is the interest of \$889·62½, for 11*mo.* 17*da.*, at 5 per cent, per annum?
4. What is the interest of \$2081·935, for 1*yr.* 10*mo.* 15*da.*, at 7 per cent, per annum?
5. What is the interest of \$5817·219, for 3*yr.* 3*mo.* 29*da.*, at 8 per cent, per annum?
6. What is the interest of £430 6*s.* 6*d.*, for 2$\frac{1}{13}$*yr.*, at 5 per cent, per annum?
7. What is the interest of £676 13*s.* 10½*d.*, for 7*mo.* 27*da.*, at 3½ per cent, per annum?
8. At \$6 per cent interest, and \$43·15 the half-yearly dividend, what is the principal?
9. At \$10 per cent interest, as in Louisiana, and \$325·171, the half-yearly income, what is the principal?
10. At £3½ per cent interest, and £69 6*d.* the half-yearly dividend, what is the principal?
11. What principal will produce an income of \$823, at \$6 per cent interest?
12. What principal will produce a half-yearly dividend of £37, at 3 per cent interest?
13. What principal will amount to \$3000, in 21 years, at \$5¾ per cent, annual interest?
14. What principal sum will amount to £1000, in 17 years, at £4½ per cent interest?
15. What will be the amount of \$3319·703, in 9 years, at \$6 per cent, annual interest?
16. What will be the amount of £805, in 3$\frac{1}{3}$ years, at £4½ per cent, annual interest?
17. When will \$919·777 amount to \$2000, at \$6 per cent, annual interest?
18. When will £1505 11*s.* 3½*d.*, amount to £2000, at £3½ per cent, annual interest?

19. What is the interest of \$666·17, from 4th March to 17th July of the same year, at \$6 per cent interest?

20. What is the interest of £149 2*s*. 11*d*., from 7th October to 3d February following, at £5 per cent, annual interest?

21. What was the interest on \$4054·18¾, from 28th January to 29th May, 1832, at \$6 per cent, per annum?

22. What is the interest of £500 for 133 days, at £3½ per cent, per annum?

23. What is the interest on \$618·12½, for 277 days, at \$6 per cent, per annum?

24. At what rate of simple interest will \$600 amount to \$1100 in 15 years?

25. At what rate of simple interest will £500 double itself in 17 years?

26. What is the interest of \$553·192, for one day, at \$7 per cent, per annum?

27. When will \$1000 double itself, accumulating at the rate of \$6 per cent, per annum, simple interest?

28. A note for \$800, bearing interest at \$6 per cent, per annum, was due 18 months from the date thereof; when a fourth of the time had elapsed, a fifth of the debt was paid, with interest; at the expiration of half the remaining time, half of the debt remaining was paid, also with interest; what sum was due at the expiration of the term prefixed?

Discount proper.

How has discount been defined by us? — *Discount proper is an allowance made on the payment of money before it is due, less than interest, but proportioned, after the same manner, to the use of money for a time.*

On what ground is discount made less than interest? — Discount in hand may itself make interest to the time originally fixed for payment of the debt.

Then what is lost by payment anticipated? — Money in possession is supposed to be making interest, and commonly does make it; a debt, therefore, paid before time, occasions a loss to the debtor of the hire of his money to the period first appointed for payment.

What is the extent of this loss? — The extent of this loss is total, on interest, if no allowance be made; but if interest be allowed him, he is a gainer, if discount, he is not a loser, by payment anticipated.

How does this appear? — If interest be allowed him on payment anticipated, by using it to the end of the original

term, he will have made more interest than he would have done, had he retained his money; for, by the supposition, he obtained the full interest on his money, at the time of making the payment.

How is this inequality obviated? — Discount is the just allowance on payments before time.

Is it the allowance usually made? — Discount proper is taken by commission merchants and brokers, on remitting money to their principals before it is due; seldom, it is believed, in other cases.

What then is done? — Interest is more commonly obtained than discount.

Does the definition apply to bills discounted, as the term is? — The advancing of money on personal security, pledged in bills for repayment at a time fixed, and not legally demandable before that time, seems to place the lender in the situation of a person paying a debt before time, and to entitle him to discount; that is, to an allowance in hand.

Which of the two allowances is actually taken? — Interest is the allowance actually taken.

Might not this be usury? — Though exceeding the legal rate of interest, courts of law have decided that it is not usury.

On what ground? — On the ground, that the excess is a very small matter, such as the law does not look to, arising out of calculations too minute for men in general to be supposed conversant with; in semblance, it may be added, not unreasonable, considering how seldom interest can be made on sums so small as those of interest itself usually are.

Do we never hear of discounts much exceeding the legal rate of interest? — Discount for prompt payment of debts, if it exceed the usual rate of interest, is manifestly so much granted to the debtor; and a man has a right to take from his claims whatever he pleases.

Would a debtor be injured by allowance of exact discount only? — On the contrary, if we look only at the hire of money, debtors are benefited by the allowance either of discount or interest on payment anticipated; for thus they realize an advantage in hand.

What are the terms in discount? — In discount proper, the only other peculiar term is present worth.

What is meant by it? — Present worth is so much of a debt as, paid before it is due, and put to interest for the period intervening, will earn a sum equal to the discount allowed for prompt payment.

Why should it earn no more? — It cannot earn interest of the debt, for it is less than the debt, by subtraction of the discount.

How is the adjustment made? — If a year's interest be added to $100, and their sum be imagined a debt proposed to be paid a twelvemonth in advance, the discount on that payment will be the rate of interest exactly; for the $100, put out to interest immediately, will have earned as much, when the term, originally fixed for payment, shall come round.

What is the application? — Therefore, in every question of discount, the hundred, added to its interest for the time expressed will form an article, and the interest itself an estimate, by which other sums and their discount may be proportioned.

Concisely, what is the present worth? — Present worth is the debt, less the discount.

Add the two together, what may the sum be called? — The present worth, added to the discount, may be called the amount; for the present worth is a principal sum proportioned to earn the discount, as interest; and principal united with interest make what is called an amount in banking transactions.

In every case of discount or present worth sought, what is *a r*? — Whenever discount or present worth is required, the article requiring estimate is, of course, the sum to be discounted.

When the present worth is sought, what will be *e g*? — Present worth being sought, the present worth given will be a hundred; for a hundred is the present worth of a hundred, plus the interest for the rate and time expressed.

How many statements are required in this rule? — In the solution of every question of discount, two statements are requisite; one, to determine the interest of a hundred for the rate and time expressed; a second, to proportion the discount sought from the interest thus found.

By what rule is discount governed? — The prior step, as in every case of interest to be ascertained, is governed by conjoint proportion; the discount is found by single proportion, there being no longer any question of time.

What prevents the use of the hundred and its annual interest in all these cases? — The hundred and its annual interest can be used only in cases where the discount sought is of a year or years; for in every case the interest added to the hundred must be of the same rate and time with the discount sought; except where principal and time of both articles are, *respectively*, measures or multiples one of the other.

Can you demonstrate this from example? — Let the rate be $6 per cent, per annum, the debt, $103, and the time anticipated, 6 months; we know that the discount of this sum is $3, because $100 in 6 months will earn that sum; yet proportioned to $106, in a conjoint statement, as for a year, the estimate found will be less than $3.

$$\frac{103 \times \frac{1}{2}\text{yr.}}{106 \times 1\text{yr.}} \text{ of } \$6 = 106 : 309 = \$2{\cdot}915\tfrac{10}{106}.$$

To what is this owing? — In proportion, the ratios given and found are equal; the ratio given in the example I have recited is $\frac{3}{103}$; consequently, the ratio found cannot be $\frac{3}{103}$, since this is little more than one half the former; it is evident also, that 106 in one year must earn more than double the interest which 103 would in half a year.

Why do you speak of doubling the interest, when we are on the subject of discount? — A statement, in conjoint proportion, of the terms in a question of discount, becomes an interest statement; but interest is more than discount, the statement therefore is erroneous and so is the result.

How happens it then, that the result should be smaller? — Interest thus found will be a trifle less than the discount sought, because the rate, or interest of the article estimated, is stated below the supposition; namely, as interest of the hundred, plus the rate; whereas the rate is of a hundred only.

What are the maxims?

MAXIMS IN DISCOUNT PROPER.

Discount sought is, as the amount of a hundred for the rate and time expressed, to the interest itself of a hundred for the same.

Proof, equality of the discount with the interest of the present worth.

Present worth sought is, as the amount of a hundred for the rate and time expressed, to a hundred, the present worth given.

Proof, equality of the present worth found, plus interest for the time, to the amount of the debt.

APPLICATION.

1. What is the discount on $100, paid 9*m.* 3*da.* in advance; allowing interest at the rate of $6 per cent, per annum?

$\frac{9m.\ 3da.}{360}$ of $6, int. of $100 for a year; $=\frac{273}{60}=$ $4·55.

The discount sought is the ratio,

$\frac{100}{104 \cdot 55}$ of $4·55, int. of 100; $=\frac{455}{104 \cdot 55}=20 \cdot 91 : 91 \cdot 00=$ $4·35198 [Ans.

	$100	amount.
	4·35198	discount.
	95·64802	pres. worth.
	·06	
$6m.=\frac{1}{2}yr.$	5·7388812	int. for a year.
$3m.=\frac{1}{2}$ of $\frac{1}{2}$	2·8694406	
$3da.=\frac{1}{30}$ of $\frac{1}{4}$	1·4347203	
	·04782401	
	4·35198491	proof.

2. What is the present worth of $365, payable $2\frac{1}{2}$ years hence, reckoning interest at $6 per cent, per annum?

$6 $\times 2\frac{1}{2}yr.=$ $15, int. of $100.

The present worth sought is the ratio,

$\frac{365}{115}$ of $100 p. w. given; $=1 \cdot 15 : 365=$ $$317 \cdot 3913\frac{1}{23}$ answer.

$$317 \cdot 3913\frac{1}{23}$	pres. worth.
·06	
$19 \cdot 043478\frac{6}{23}$	int. for a year.
2·5*yr.*	
$95217396\frac{12}{23}$	
38086956	
$47 \cdot 6086956\frac{12}{23}$	int. for $2\frac{1}{2}yr.$
$317 \cdot 3913000\frac{1000}{23}$	pres. worth.
$365·000000	proof.

Examples to be wrought, proved, and recited.

1. What is the discount on \$145·16, payable 3*m*. 3*da*. hence; at \$6 per cent interest?
2. What is the present worth of \$780·455, payable 5 months hence; at \$6 per cent interest?
3. What is the discount on \$1000, payable 297 days hence; at \$7 per cent interest?
4. What is the present worth of \$690·13$\frac{7}{8}$, payable 364 days hence; at \$8 per cent interest?
5. What is the discount on £1200 3*s*. 4*d*., payable 7 months hence, with forbearance; at £5 per cent interest?
6. What is the present worth of £84 5*s*. 1$\frac{1}{2}$*d*., payable 9 months hence, with forbearance; at £4$\frac{1}{2}$ per cent interest?
7. What is the discount on \$337, payable 2*yr*. 2*mo*. hence; at \$6 per cent interest?
8. What is the present worth of \$6001·631, payable 3*yr*. 3*da*. hence; at \$7 per cent interest?
9. What is the discount on a debt of £500, payable, one half, 117 days hence, one third of the remainder, 7 months hence, and the residue, 15 months hence; reckoning interest at £3$\frac{1}{2}$ per cent, per annum?
10. What is the present worth of \$1465·819, payable, in equal portions, at 5 months, 11 months, and 19 months; reckoning interest at \$7 per cent, per annum?
11. What is the difference between interest and discount on \$1000, payable 11 months hence, with forbearance; at \$6 per cent interest, per annum?

Compound Interest.

Do you recollect what compound interest was said to be? — *Compound interest is interest calculated on the principal sum due, and on the continually accruing interest, unpaid, and added to the principal.*

Is compound interest legal? — Compound interest, I believe, is nowhere legal; yet there are practices allowed which amount perhaps to the taking of compound interest.

How often is interest usually payable? — On promissory notes interest is commonly payable every two or three months; in most other cases, every six months; rarely at the end of twelve months.

Are these periods never made shorter? — Usurers often make their exactions every month.

What is the object of this? — The quicker the return of interest, the greater is the compound interest; not less by laying out money anew, than by calculating interest anew.

Can you give an example? — If $100 interest be payable at the end only of a year, no compound interest can accrue till after the expiration of the year; but $50, paid half yearly, may earn half a year's interest on itself, by the end of the year.

In what transactions does compound interest itself seem to be taken? — On bills not bearing interest till over due, interest calculated thereafter is usually added to the principal, and a balance struck, every time that payment in part is made.

Is there any foreign instance? — Compound interest is charged in Britain, in much the same manner, on mercantile balances unsettled, and carried from one year's account to another.

What is the best mode of realizing compound interest? — An unobjectionable mode of realizing compound interest is the speedy reinvestment, on good security, of funds as they come in.

Is compound interest producible by a single statement? — In compound interest there must be as many statements as there are periods of interest due; because every addition of interest to a former principal creates a new principal sum, the interest of which is to be calculated separately.

How is the principal found? — The principal must be sought as under simple interest, from the amount, the rate, and the time.

What is the process? — By finding the amount of a hundred, at compound interest, for the time given, a ratio may be formed with the amount requiring estimate; and the principal being sought, a hundred will be the principal given.

How is the time found? — By raising the principal to the amount, and at the rate, specified, we shall learn the number of interest periods; should the last be in excess of the amount, by proportioning that excess to the interest of the last period, the estimate found will be the excess of time; which, subtracted from the sum of the periods found, will give the exact time.

What are the maxims?

MAXIMS IN COMPOUND INTEREST.

Compound interest sought is the sum total of simple interests, calculated, first, on the principal; then, on every successive amount, at every return of interest due and unpaid; the last product being the total amount of the principal specified.

Proof, from the amount obtained, to find the principal.

Principal sought at compound interest is, as the amount of a hundred at the same to a hundred, the principal given.

Proof, from the principal obtained, to find the amount.

Time sought at compound interest is the number of interest periods within which the principal will be raised to the amount specified, less any excess of time, to be proportioned from the last period.

Proof, from the time obtained, the amount, and the rate, to find the principal.

APPLICATION.

1. What will be the compound interest and amount of $6028·19, in 2*yr.* 2*mo.*, at $6 per cent interest, per annum, payable half yearly?

To find the interest.

$\$2=\frac{1}{50}$ of 100	$6028·19	principal.
$1=\frac{1}{100}$	120·5638	
	60·2819	
$2=\frac{1}{50}$	6209·0357	2d h. y. prin.
$1=\frac{1}{100}$	124·180714	
	62·090357	
$2=\frac{1}{50}$	6395·306771	3d
$1=\frac{1}{100}$	127·906135	
	63·953067	
$2=\frac{1}{50}$	6587·165973	4th
$1=\frac{1}{100}$	131·743319	
	65·871659	
$1=\frac{1}{100}$	6784·780951	2*m.*'s prin.
	67·847809	
	$6852·628760	amount. Ans.
	6028·19	deduct prin.
	$824·43876	comp. int. Ans.

2. To find the principal.

$$\begin{array}{rl}
\$100 & \\
3 & \text{1st half-year's rate.} \\
\hline
\cdot 03\times 103 & \text{2d h. y. principal.} \\
3\cdot 09 & \\
\hline
\cdot 03\times 106\cdot 09 & \text{3d.} \\
3\cdot 1827 & \\
\hline
\cdot 03\times 109\cdot 2727 & \text{4th.} \\
3\cdot 278181 & \\
\hline
\cdot 01\times 112\cdot 550881 & 2m.\text{'s} \\
1\cdot 12550881 & \\
\hline
\$113\cdot 67638981 & \text{amount of 100.}
\end{array}$$

The principal sought is the ratio,

$\frac{6852\cdot 62876}{113\cdot 676389}$ of \$100, prin. given; $=113\cdot 676389 : 685262\cdot 876 =$ [\$6028·2 proof.

3. To find in what time, \$6028·19 will amount to \$6852·62876, at \$6 per cent, compound interest, payable half-yearly.

The work being the same through the first 4 periods, as in obtaining the answer above, will be continued thus:

$$\left\{\begin{array}{l} 2=\frac{1}{50} \\ 1=\frac{1}{100} \end{array}\right. \left|\begin{array}{r} 6784\cdot 780951 \\ 135\cdot 695619 \\ 67.847809 \\ \hline 6988\cdot 324379 \\ 6852\cdot 62876 \\ \hline 135\cdot 695619 \end{array}\right. \begin{array}{l} \\ \\ \\ \\ \\ \text{amount.} \\ \\ \text{excess of interest.} \end{array} \qquad \begin{array}{r} \$135\cdot 695619 \\ 67\cdot 847809 \\ \hline 203\cdot 543428 \end{array} \text{ int. of 5th [period.}$$

The excess of interest in this case being exactly that given by $\frac{2}{3}$ of the $\frac{1}{2}$ year's rate, we might infer that the excess of time is the same; but for reference in dissimilar cases, we shall complete our exemplification of the rule.

The excess of time is the ratio,

$\frac{135\cdot 695619}{203\cdot 543428}$ of $\frac{1}{2}y.$, t. giv.; $=203\cdot 543428 : 67\cdot 8478095 = \cdot\dot{3}y.$

$$\begin{array}{rl}
2yr.\ 6m. & \text{5 periods.} \\
4m. & \text{excess.} \\
\hline
2yr.\ 2m. & \text{ans.}
\end{array} \qquad \begin{array}{r} 12m.=1y. \\ \hline 4\cdot 0 \end{array}$$

Examples to be wrought, proved, and recited.

1. What is the compound interest on \$895·13, forborne 3*yr.* 7*mo.*, at \$6 per cent, per annum?

2. What is the amount at compound interest, of \$1000, forborne $2\frac{1}{2}$ years, at \$6 per cent, per annum, payable half-yearly?

3. What is the compound interest on £760 17*s.* 8*d.*, forborne $1\frac{3}{4}$ year, at £$4\frac{1}{2}$ per cent, per annum, interest payable quarterly?

4. What is the amount, at compound interest, of £3011 7*s.* $6\frac{1}{2}$*d.*, forborne $2\frac{1}{2}$ years, at £$3\frac{1}{2}$ per cent, per annum, interest payable half-yearly?

5. What will be the compound interest due on \$623·5, from January 9, 1832, to November 11, 1833, at \$7 per cent, per annum, payable half-yearly?

6. What is the amount, at compound interest, of \$336·814, forborne from October 3, 1829, to July 8, 1832; at \$8 per cent, per annum?

7. What principal sum will amount to \$1000 in 5 years, reckoning compound interest, at \$5 per cent, per annum?

8. What principal sum will amount to \$1575·111 in 3 years, reckoning compound interest, at \$6 per cent, per annum, payable half-yearly?

9. In what time will \$416 amount to \$600, reckoning compound interest, at \$7 per cent, per annum?

10. When will \$1000 double itself, at compound interest of \$$4\frac{1}{2}$, \$5, and \$6, per cent, per annum?

Compound Discount.

What is compound discount? — Compound discount is discount proportioned to compound interest, as simple discount is proportioned to simple interest.

Of what calculations is it an element? — Compound discount is chiefly resorted to in calculating the value of annuities for a limited term of years.

How is it applied? — As the article estimated in simple discount is the hundred, plus the simple interest, in compound discount it is the hundred, plus the compound interest.

Why is it applied? — It is applied to equalize advantage on the part of vendor and vendee, of payer and receiver.

Can you exemplify this? — The present value of \$112, payable two years hence, is not \$100, although, at \$6 per cent interest, \$12 are the simple discount for that time.

Why should it not be the value? — Because there is a loss

to one party, and a gain to the other, of interest upon interest; namely, on \$6, during a whole year, a difference that amounts to \$·36; therefore the article estimated ought to be, in the case supposed, \$112·36; that is, the hundred, increased by two years' compound interest.

·06×106	2d yr.'s prin.
6·36	
\$112·36	

In making up compound interest of the hundred, how must the time be taken? — The time, in all such calculations, should be divided into interest periods; these will never be fewer than the number of years to the time when the debt is due; usually, as many half-years; for interest is, in general, paid half-yearly.

What are the maxims?

MAXIMS IN COMPOUND DISCOUNT.

Compound discount sought is, as the amount of a hundred at compound interest, for the rate and time expressed, to the compound interest itself, the estimate given.

Present worth at compound discount is, as the amount of a hundred at compound interest, for the rate and time specified, to a hundred, the present worth given.

Proof, equality of the present worth found, plus the compound interest for the time, to the amount specified.

APPLICATION.

£100 are payable one year hence, and £105, three years hence; what is the present worth of these debts, at compound discount, allowing £5 per cent, annual interest?

First instalment.

The present worth sought is the ratio,

$\frac{100}{105}$ of £100, p. w. given; $=1·05 : 100·0=£95·\dot{2}3809\dot{5}$.

	£100	
	5 rate.	
£5=$\frac{1}{20}$ of 100	105	2d yr.'s prin.
	5·25	
5=$\frac{1}{20}$	110·25	3d
	5·5125	
	£115·7625	amount of 100 in 3 years.

Second instalment.

The present worth sought is the ratio,

$\frac{105}{115\cdot7625}$ of £100, p. w. given; $=\frac{105}{115\cdot7625}=\frac{21}{\cdot231525}$

$=\cdot077175 : 7\cdot00000 = £90\cdot70294$

£95·238095 p. w. of 1st inst.
90·702947 ——— 2d

185·941042

18*s*. 9¾*d*.

£185 18*s*. 9¾*d*. answer.

Rate of unity ·05 × £185·941 p. worth.
9·29705

·05 × 195·23805 2d yr.'s prin.
9·7619025

Amount £204·9999525 proof.

The proof never can be exact, because the first p. w. is a repetend, and the second a simple approximate. If the discount had been sought in the first instance, *e g* of the 1st instalment would have been £5; of the 2d instalment, £15·7625.

If the question were now put, in what time will the present worth thus found amount to £205, we perceive from the proof, that the number of interest periods are two precisely, if considered annual; that the time is therefore, two years. Thus advantage is equalized between debtor and creditor, wherever compound interest may arise; and it must arise whenever more than a single customary interest period intervenes between a time present, and any future date fixed for the discharge of a debt; the compound discount allowed to the debtor is equal, exactly, to the compound interest that may accrue to the creditor. Attending to the result in the present case, this is manifest almost without computation; for £100, retained during a year longer than originally agreed upon, will earn £5 interest at the supposed rate; and the discount of £105, paid a year in advance, is also £5; both therefore are equal; and the method by compound discount and compound interest is shown to give the true intermediate time for the simultaneous discharge of instalments. In the present case, the time found by simple discount and interest would be 8 days short of 2 years; and if we compute the time as in simple interest, after subtracting the present worth obtained at compound discount from the amount of the instalments, there is an excess of 18 days over the 2 years; both manifestly erroneous periods; a third, not less so, will be

found under equation of payments, where the computation is by simple interest, alike for the time gained and lost. The example has been taken from Dr. Webber, who recommends a rule of Malcolm's, just in itself, as appears by the result, but incomprehensible to any one who is not a profound mathematician. The Doctor himself falls into the error of supposing the loss on payment in advance to be of discount only.

Examples to be wrought, proved, and recited.

1. What is the present worth of \$456, payable 3 years hence, at compound discount; reckoning interest at \$6 per cent, per annum?

2. What is the compound discount on \$2000, payable 4¾ years hence, at \$7 per cent, per annum?

3. What is the present worth, at compound discount, of £761 1*s.* 1½*d.*, payable 3 years hence; reckoning interest at £3½ per cent, per annum?

4. What is the present worth of £375 12*s.* 7¾*d.*, payable 2¼ years hence, at £4½ per cent interest, and allowing compound discount?

5. What is the compound discount on \$633·33, payable 1½ year hence, and \$366·67, payable 2½ years hence, at \$5 per cent annual interest?

6. What is the present worth, at compound discount, of a debt payable in two instalments; namely, \$881·913, two years, and \$118·087, three years and 5 months hence; at \$8 per cent annual interest?

Equation of Payments.

Instalments not equated by interest only.

What is the effect of payment deferred? — A debt, not paid at the time stipulated, or justly reckoned upon, gives to the debtor, for all the time deferred, interest appertaining to the creditor.

What of payment anticipated? — A debt paid, without deduction, before it is due, occasions a loss to the debtor, of interest for all the time anticipated.

Is it not discount that he loses? — If an allowance were made, he would be entitled to discount, and on the discount itself he might make interest; therefore, if allowed nothing on payment anticipated, he loses more than discount; that is, he loses interest.

The controversy that has existed on the subject of equation of payments seems to have originated entirely from writers overlooking the distinction between what a debtor who anticipates the period of payment is entitled to receive, and what he would lose if he received nothing.

Why is he not entitled to interest? — The debtor is not entitled to interest, because the creditor cannot, with a principal diminished by interest, make a sum equal to the interest allowed, by the time the debt shall become due, as originally agreed upon.

If a debt be owing, payable by instalments, what will be the effect of discharging it in a single sum, at an intermediate time? — The paying of instalments together, without deduction, at an intermediate time, gives to the debtor interest on the money first due, from the stipulated to the intermediate time; with interest on that interest, from the intermediate to the farthest time originally designed.

What does it take from him? — It takes from him interest on remoter instalments during the same second part of the interval.

Whom would such an arrangement most benefit? — The debtor will be most benefited by an arrangement of the time that shall make the interest on the advanced payments equal to the interest on the deferred.

Why so? — Because, on the interest of the payments deferred he may gain compound interest to the time of final discharge originally stipulated; whereas the creditor, though receiving the debt undiminished, has, at the time of receiving it, no interest in hand on which to make compound interest.

How might the advantage be equalized? — Since a debtor is entitled only to discount on money paid in advance, if an intermediate time of general discharge is to be fixed, he cannot be injured by proportioning the time partly to discount.

Will this produce parity? — This will produce perfect equality of advantage, if the interest on payments postponed be made equal only to the discount on payments in advance; for equal sums in the hand of debtor and creditor, employed during the same period, may produce equal interest.

But have you not supposed the debt to be paid in full? — Although no deduction be made from the debt at the time equated, the debtor has already the interest on payments postponed, or has had the use of the money, which is considered the same; the creditor keeps the discount on the payments in advance, and thus the advantage is equalized.

Supposing these nice calculations to be inconvenient in business, what must be done? — If the equating of payments by compound discount be too complex for general business, and a trifling advantage in time, by calculating interest instead of discount, be not inequitably given to a debtor, we may con-

tent ourselves with the common rule, which computes interest on both sides.

What is paid at a time of general discharge? — At an equated time, the whole sums due must be discharged, without deduction, allowance, or forbearance whatever; for the time computed is made a perfect, or more than perfect compensation.

Can you now define the rule? — *Equation of payments is the proportioning of an intermediate time, for the simultaneous discharge of debts payable by instalments, so as to equalize interest gained with interest or discount lost.*

Demonstration of rule.

How may interest on different sums be equalized? — If a sum be lent as much longer or shorter a time, as it is less or greater in amount than another, their interest at the same rate will be equal, the difference of amount being set off by time.

Under what rule would such a transaction fall? — Equalization of interest by sums of unequal amount falls under the rule of proportion inverse; for the greater principal of two must be kept the shorter time, the smaller principal the longer time, to equalize interest.

How would interest be calculated on a number of distant payments? — Interest is calculated, in every case, by multiplying principal and time into the rate of unity.

Where no interest is payable, what grounds of calculation have we? — Should interest not be payable, the productiveness of capital may be calculated on the supposition that interest is made; for wherever vested, capital does ordinarily make interest.

When therefore the rate, real or supposed, is the same, and an intermediate time of payment is sought, what is the proceeding? — To find an intermediate time for the simultaneous discharge of distant instalments, computing simple interest at the same rate, the sum of the products of every instalment, multiplied into its time, must be divided by the sum itself of the instalments.

How is this demonstrated? — The question being, at what time? three things are known; the debt, or article requiring estimate; the instalments, or article estimated, for their times are known; and consequently, the times themselves; and the time sought is one at which the whole debt can be discharged on terms of equal advantage with the discharge of the instalments severally.

What is the inference? — The question is therefore one of proportion inverse; for the greater the debt, the shorter is the time, the smaller the debt, the longer is the time, required to produce interest previously limited.

What limitation of interest is there in the case? — The limitation of interest is in the productiveness of the instalments; for a time is sought at which the productiveness of the debt, in one entire sum, shall be equal to that of its portions discharged at distant periods; reckoning the interest, on either side, to the completion of the last period.

Can you now apply your demonstration to the rule? — In proportion inverse, the article requiring estimate is the denominator; and the rule makes the sum of the instalments the divisor; the article estimated, or instalments, is the numerator; and this is multiplied into the times of the instalments, or estimate given.

What then is the statement? — The statement, in equation of payments, if drawn out in form, is that of a collective ratio, the numerator of which contains the instalments, each separately multiplied into its appropriate time.

Where is *e g*? — The times cannot be stated in the usual form of an estimate given, because they constitute distinct members, distinctly to be multiplied into corresponding members of the article estimated.

Why may not the members be added together? — Should the members of either term be added together before their distinct and separate multiplication, the productiveness of the instalments would be enormously falsified; for each instalment would thus be multiplied into its own time, and also into the time of every other.

Can you follow up your demonstration with an example? — Let \$50 be payable at 6 months, another \$50, at 9 months, hence; the sum of the products of the instalments and times is 750: this, divided by the debt, gives $7\frac{1}{2}$ months as the equated time.

Rule.

$$
\begin{array}{lcl}
\$50\times 6m. & = & 300 \\
50\times 9 & = & 450 \\
\hline
100 & : & 750 = 7{\cdot}5m.\ \text{ans.}
\end{array}
$$

How do you show it to be in proportion inverse? — \$100, the debt, is made the denominator, the two fifties, multiplied, each into its time, the collective numerator, of the ratio.

Collective inverse ratio.

$$\frac{\overline{50\times 6m.}+\overline{50\times 9m.}}{100}=7{\cdot}5m.$$

How do you show its correctness? — At \$6 per cent interest, it is manifest that \$50 will, in half a year, earn \$1½; in 9 months, \$¾ more, or \$2·25; therefore the two instalments would earn \$3·75; and the same is the product of 7½ months, multiplied into the rate, or interest of \$100, the debt.

	Interest.	
$6m. = \frac{1}{2}yr.$	\$6	int. of 100.
$1\frac{1}{2}m. = \frac{1}{4}$ of $\frac{1}{2}$	3	
	·75	
	3·75	proof.

On the nature of the rule for the equation of payments, as exhibited in the preceding demonstration, the author, unable to collect a glimmering of light from any book that has ever fallen in his way, or which he has ability to consult, has been left entirely to his own investigations.

Can you now give me the maxims?

MAXIMS IN EQUATION OF PAYMENTS.

Equated time of instalments, computed at simple interest, is, as the debt to the sum of the products of every instalment, multiplied into its time.

Proof, equality of interest gained and lost.

Time to be equalized on part payments advanced and postponed is found and proved by the rule of proportion inverse.

True equated time is that period when the present worth of instalments, taken at compound discount, will, at compound interest, amount to the debt.

APPLICATION.

1. £100 are payable one year hence, and £105, at 3 years hence, interest reckoned at £5 per cent; what would be the equated time for the simultaneous discharge of both instalments?

$100 \times 1yr. = 100$ product of 1st instalment.
$105 \times 3 \quad = 315$ ———— 2d

$205 \quad : \quad 415 = 2\cdot\dot{0}243\dot{9}yr.$
$12m. = 1yr.$

29280
$30da. = 1m.$

Answer, 2*yr.* 8*da.* 8·784

Collective inverse ratio,

$$\frac{\overline{100\times1yr.}+\overline{105\times3yr.}}{205}=2\cdot\dot{0}243\dot{9}yr.$$

$1\cdot\dot{0}243\dot{9}yr.$ postponement of 1st instalment.
5 rate of 100.

£5·12200 gain to the debtor.

$1\ \ \ \ yr.$
$\cdot\dot{0}243\dot{9}$

·97562 anticipation of 2d instalment.
105 2d instalment.

487810
97562

102·44010
·05 rate of unity.

£5·122005 interest lost to the debtor. Proof.

The fraction of a day must be reckoned favorably to the creditor; for we have shown, under compound discount, that the true equated time in the present case is two years only; and both there, and under the present rule, that in every case, the equating of payments at simple interest is an advantage to the debtor.

2. $240 were payable at the expiration of 6 months; but, in $1\frac{1}{2}$ month, $60 were paid; in $4\frac{1}{2}$ months afterward, $80 more; how much longer than the 6 months might payment of the remainder be deferred, as a just compensation?

$60 in advance, $4\frac{1}{2}m.$
80 in season.
100 postponed.

$240 debt.

The postponement sought is the inverse ratio,
$\frac{60}{100}$ of $4\cdot5m.$, time anticip.; $=\frac{27}{10}=2\cdot7m.$

$30da.=1m.$ $\left\{\begin{array}{l}\frac{6}{10}\\ \frac{2}{4}\cdot\frac{7}{5}=\cdot6 \text{ proof.}\end{array}\right.$

Answer, $2m.$ $21da.$ 21·0

Proof by equality of interest.

$$
\begin{array}{r|l}
 & \$60 \\
 & \cdot05 \text{ rate of unity.} \\
\hline
4m.=\frac{1}{3}yr. & 3\cdot00 \\
\hline
\frac{1}{2}m.=\frac{1}{8}\text{ of }\frac{1}{3} & 1\cdot \\
 & \cdot125 \\
\hline
 & \$1\cdot125 \text{ int. of } \$60 \text{ during } 4\tfrac{1}{2}m. \\
\hline
2m.=\frac{1}{6}yr. & \$5 \qquad \text{rate of 100.} \\
\hline
20da.=\frac{1}{3}\text{ of }\frac{1}{6} & \cdot8333\dot{3} \\
1da.=\frac{1}{20}\text{ of }\frac{1}{18} & \cdot2777\dot{7} \\
 & \cdot0138\dot{8} \\
\hline
 & \$1\cdot12500 \text{ int. of } \$100 \text{ during } 2\cdot7m. \\
\end{array}
$$

The proof by equal ratios shows that the work is right; the proof by interest further demonstrates, that our supposition of inverse proportion, and the maxim built upon it, are right also.

Examples to be wrought, proved, and recited.

Interest at 6 *per cent.*

1. \$316·37½ are payable in two instalments, the first of \$150, at 6 months, the remainder at 12 months; at what equated time should the whole be paid?

2. \$748 are payable in three instalments of equal amount; the first, 9 months hence, the others at the same interval, in succession; what would be the equated time?

3. \$35 are payable 3 months hence, \$68 at 6 months, and \$79·229 at 15 months; at what distance of time might the whole be paid without injury to either party?

4. \$1000 are payable on three notes of equal value, in successive periods of 9 months; what would be the equated time for paying the whole?

Interest at 5 *per cent.*

5. A debt is payable, $\frac{1}{8}$ at the expiration of 3 months, $\frac{1}{8}$ at 9 months, $\frac{1}{4}$ at 12 months, the remainder at 2 years from the time present; what would be the equated time?

6. \$700 being payable at the end of a year, \$115·17 are advanced, by desire of the creditor, at the expiration of 3 months, and \$55·03 a month afterward; what is the equitable time for paying the remainder?

7. Three notes of \$70, \$178, and \$34, payable at intervals of 3 months in succession, are sought to be discharged presently; what sum should be paid in respect of them?

8. Three notes of \$150 each are payable, one on 21 March, another on 21 June, the third on 21 September; if discharged on 1 July of the same year, what sum ought to be paid for the three?

Valuation of Estates.

Definitions.

What may be denominated estates? — Estates consist of lands, houses, goods, money lent at interest or vested in annuities.

What is an annuity? — An annuity is a payment, to be made yearly, of a sum fixed, during a limited number of years, or during life, or indefinitely, till the repayment of the purchase money.

Are there none called perpetual annuities? — Some annuities for an indefinite term are called long, or perpetual, annuities; freehold lands and houses, though subject to variable rents and deficient payments, are commonly valued as perpetual annuities.

Of all these, what are the subjects of valuation by arithmetical rule? — Limited and perpetual annuities are strictly subjects of arithmetical valuation; life annuities involve considerations peculiar to the subject.

Are there any means of determining the average length of a life? — The computed average length of life, as applied to large classes of men, depends on the consideration chiefly of age and locality; as applied to individuals, it depends also on health, character, and occupation.

What results from the computation? — When the average length of a life has thus been calculated, the value of an annuity dependent on it may be estimated as for a limited term, which may fail individually, either in excess or defect, but will be true in the mass, or great number of similarly circumstanced lives reckoned upon to form the average.

By what rule are estates valued? — Perpetual annuities, that is, most estates, are valued by the rule of single proportion direct.

Not by some rule of interest? — An annuity is itself interest or income, arising from the investment of capital, either moneyed or real.

What is meant by real, in these distinctions? — *Real*, as applied to estates, signifies property immovable, or land and houses; perhaps designated as real, because supposed to possess more of reality, than the shifting nature of money and goods seems to claim.

What are the latter species of property denominated? — Money and goods are called personal estate, because they are removable with the possessor.

Is there not still another description of property? — Leasehold estate is property holden for a term, fixed by a writing called a lease, and conditioned on the payment of a rent.

Are such never considered annuities? — Leasehold estates may be considered and valued as annuities for a limited term, provided the reserved rent be nominal, as sometimes is the case; and to the extent of the difference, if less than a usual rent.

Principles of valuation.

According to what mode of estimate are estates bought and sold? — Annuities proper are estimated at a rate per cent; but the valuation of land and houses at a per-centage is personal only to the party making it, or is merely hinted at in conversation.

Of what use is such a valuation? — The mental valuation of estates at a per-centage is of great weight in determining buyer and seller; since in judging of a price offered or asked, parties will consider the probable income to arise from the purchase or price, and will compare it with that arising from other modes of investment.

Estates in fee and perpetual annuities.

In valuing an estate, what will be *a e*? — The value of a freehold estate, at a per-centage, is, as the rate of a hundred, to the income actually arising from the estate; for the rate is the income from a hundred, and will therefore be the article estimated, a hundred being the estimate given.

Has compound interest a place in such calculations? — A freehold estate in possession is equivalent to capital in hand; a perpetual annuity, in due course of payment, is simple interest continually arising on vested capital; therefore compound interest has no place in such valuations.

What is to determine the rate? — In purchasing annuities, the rate must be fixed, by agreement between the parties; in

valuing estates, a party will probably make his estimates of purchase and price at different rates, for the sake of comparison; for income and interest often differ very widely on the same reputed amount of capital.

Owing to what? — It is owing to the different estimation which attaches to different kinds of property, land being most valued in some places, cash in others; in general, the income from land is much smaller than interest of money; and in many parts of the U· S· is little more than nominal.

Limited annuities.

By what rule are annuities for a term of years valued? — Limited annuities are valued by the rule of compound discount.

In simple discount, is either party favored? — Discount proper must have been introduced as a fair and equal arrangement between two parties; and in calculation it is so, exactly.

Is it otherwise in practice? — As it is much easier to realize interest on large than on small sums, the holder of a small discount seems in this respect least favored.

In whose situation, of the payer and receiver of a bill on discount, does the purchaser of an annuity, for a term of years, stand? — The purchaser of a limited annuity stands in the situation of the payer of a bill in advance, and discounted; for he advances the purchase money.

In what do compound interest and compound discount differ? — Compound discount is the application of compound interest to debts paid in advance; and is less than compound interest, as simple discount is less than simple interest.

Then which is greatest of the two kinds of discount? — Compound discount is as much greater than simple discount as the one kind of interest is than the other.

How made so? — So made, by increasing the article estimated with interest on interest; the divisor being thus made larger, the article requiring estimate, or the thing to be purchased, gives a less amount of purchase money.

Is it lawful to take compound discount? — It is the general and allowed practice everywhere, in computing the value of annuities for a limited term of years, to calculate by compound discount.

What would be the effect of a different mode of calculation? — The disadvantage that would otherwise arise to a purchaser, in the loss of interest upon interest for years, is so great, that no one who understands the subject would buy on such terms.

How then must their value be found? — To find the present worth of a limited annuity, the present worth of every distinct payment must be ascertained, by comparing it with the amount of a hundred, at compound interest, for the same time and at the same rate.

What are the articles of the ratio? — The article estimated is the amount of a hundred up to each successive period; and the article requiring estimate is the annuity itself, or whatever portion of it is, at each successive period, to be paid; for this includes both its present worth and the discount.

What will be *e g*? — The estimate, or present worth, given will be the hundred; because that is the present worth of a hundred increased by the compound interest to the period in question.

How will the whole present worth be obtained? — The exact present worth of a limited annuity is the sum total of the present worth of all the payments, to the end of the term, severally calculated at compound discount.

Annuities and rents forborne.

How will you determine the amount of an annuity neglected to be paid? — The amount of an annuity, or rent, forborne to be paid, must be calculated at simple interest.

Why do you particularize simple interest? — Though rent be not, yet an annuity is itself, interest; therefore to calculate even simple interest upon it, has been sometimes mistaken for compound interest.

In what consists the difference? — It differs materially from compound interest; for, *in calculating interest on annuities forborne, none is calculated on the continually accruing interest.*

Are annuities paid yearly only? — Annuities and rents are commonly divided into half-yearly payments, like the dividends on stock.

What is the graduation of interest on annuities unpaid? — On the first annuity, or first portion of it forborne out of a number unpaid, interest is due for the whole number of periods of payment assigned, less one; because there is default upon it at every such period, except the last.

Why is the last excepted? — The last period of payment assigned, is considered not in default; for the amount is made up to the last, in order to a discharge.

How on succeeding portions? — On every annuity, or portion of it, due after the first, in succession, the number of defaults is one fewer than on that immediately preceding; for

interest on the second payment forborne does not arise till after the second default; and so to the last.

How may the whole number of defaults best be computed? — On the first portion due, the number of defaults is equal to the whole number of payments assigned, less one; that number diminishes by unity to the last portion, which is chargeable with none.

What is the inference? — Therefore, if we set down the whole number of stipulated payments, less one, as the first member of a series descending to unity, their sum will be the whole number of defaults; since on the last portion but one, there is only a single default.

Can you exemplify this? — If a rent, payable half-yearly, have been forborne two years, then, on the first payment due, 3 defaults are chargeable; on the second, 2 defaults; on the third, 1; in all, 6. $3+2+1=6$ defaults.

How will you make up the amount? — The amount of an annuity forborne is the simple interest accruing on every default, added to as many times the annuity, or portions of it, as are due.

Reversions.

What is a reversion? — An estate in reversion is one to be possessed, either at the expiration of a period fixed, or after the death of person or persons.

How are reversions after a term valued? — Reversions are first valued from their income, as perpetual or limited annuities; the value found is then reduced to its present worth by compound discount.

Is the same rule applicable to reversions dependent on life? — The computed average duration of lifeholders may be taken as the period of realization to a purchaser; and thus the same rule be applied.

Can you now form these observations into maxims?

MAXIMS IN THE VALUATION OF ESTATES.

The value of a freehold estate, or perpetual annuity, is the principal sum producing the income as interest, calculated at rates usual or agreed upon; and found, and proved, by the rule of single proportion direct.

The present worth of a limited annuity is the sum total of the present worths of all the payments to the end of the term, separately calculated at compound discount.

Proof. The amount of the present worth found, computed from the amount of a hundred, is equal to the amount of all

the annuities to the end of the term, computed successively at compound interest.

The amount of an annuity forborne is the simple interest accruing on every default, added to as many times the annuity, or portions of it, as are due.

The value of a reversion is the principal sum producing the income, as of an immediate estate, reduced to its present worth by compound discount.

Proof, as in proportion direct and compound discount.

APPLICATION.

1. The rent of a house is \$375; what is its value, computing interest at \$8 per cent?

The value sought is the ratio $\frac{375}{8}$ of \$100 value given; = [8 : 375=\$4687·5 ans.

$\frac{100}{8}=12{\cdot}5$. $\frac{4687{\cdot}5}{375}=12{\cdot}5$ proof.

In the valuation of land and houses, only the clear rent should be considered, exclusive of taxes and insurance. The *rate* of estimation differs widely in different places; and it must be recollected, that this mode of valuing real estate is no common ground of purchase and sale, but intended solely for personal information and direction.

2. What is the present value of an annuity of \$319, payable in half-yearly instalments, and to continue 3 years; computed at compound discount, and at the interest rate of \$6 per cent, per annum?

```
                   $100
                      3    half-yr.'s rate.
                   ----
rate of unity ·03×103    2d h. yr.'s principal.
                   3·09
              ---------
              ·03×106·09    3d
                   3·1827
              -----------
              ·03×109·2727    4th
                   3·278181
              -------------
              ·03×112·550881    5th
                   3·37652643
              ---------------
              ·03×115·92740743    6th
                   3·4778222229
              -----------------
                 $119·4052296529 amount of 100 at comp.
                 ===============                  [int.
```

$\frac{1}{2}$ of \$319=\$159·5

The p. w. of 1st half-year's instalment is the ratio, $\frac{159\cdot5}{103}$ of \$100 p. w. given; =103 : 15950=\$154·8543

103	: 15950=	\$154·8543	p. w. 1st h. y. instalment.
106·09	: 15950=	150·3449	2d
109·2727	: 15950=	145·965	3d
112·5508	: 15950=	141·7137	4th
115·9274	: 15950=	137·5861	5th
119·4052	: 15950=	133·5787	6th

Present worth \$864·0427 ans.

·03× \$159·5 1st h. y. instalment.
4·785 interest.
159·5 2d h. y. instalment.

·03× 323·785 —— principal.
9·71355 interest.
159·5 3d h. y. instalment.

·03× 492·99855 —— principal.
14·7899575 interest.
159·5 4th h. y. instalment.

·03× 667·2885075 —— principal.
20·018655225 interest.
159·5 5th h. y. instalment.

·03× 846·807162725 —— principal.
25·40421488175 interest.
159·5 6th h. y. instalment.

\$1031·71137760675 proof.

The amount sought at compound interest is the ratio, $\frac{864\cdot0427}{100}$ of \$119·4052, am. of 100; =8.640427×119·4052=\$1031·69

The excess of \$·2 in the proof will be readily perceived to arise from the circumstance of its perfect accuracy; whereas in the divisions many fractions are necessarily lost; I now regret however my not having carried the quotients to millionths, always a more satisfactory termination than any place short of them, and occasioning little more trouble. The proof is of my own devising; I can imagine no other; and though operation and proof may seem excessively long, the case itself is one of rare occurrence, and well deserving of close attention.

3. What is the amount due upon an annuity of $177, forborne during 5 years; reckoning $5 per cent annual interest?

$4+3+2+1=10$ defaults.

$177
·05

8·85 × 10 = $88·5 interest due.
177 × 5 = 885· payments forborne.

$973·5 amount. Answer.

Probably the only method of proof is to calculate apart what is due on each payment forborne.

4. What is the present worth of a freehold estate, to be entered upon 2 years hence, and producing a rent of £125, payable in half-yearly instalments; interest reckoned at £3 per cent, per annum?

The value, as immediate, is the ratio,

$\frac{125}{3}$ of £100, value given; $=3:12500=$ £$4166{\cdot}\dot{6}$

$\left\{\begin{array}{l}\frac{100}{3}=33{\cdot}\dot{3}\\ \frac{4166{\cdot}\dot{6}}{125}=33{\cdot}\dot{3}\end{array}\right.$ proof of immediate value.

£100 principal.
1·5 1st h. year's int.

$\left\{\begin{array}{l}£1=\frac{1}{100}\\ ·5=\frac{1}{200}\end{array}\right.$ | 101·5 2d h. y. prin.
1·015
·5075

$\left\{\begin{array}{l}1=\frac{1}{100}\\ ·5=\frac{1}{200}\end{array}\right.$ | 103·0225 3d
1·030225
·5151125

$\left\{\begin{array}{l}1=\frac{1}{100}\\ ·5=\frac{1}{200}\end{array}\right.$ | 104·5678375 4th
1·045678375
·5228391875

£106·1363550625 amount of 100 at comp. int.

The present worth sought is the ratio,

$\frac{4166\cdot\dot{6}}{106\cdot1363}$ of £100, p. w. given; $=1\cdot061363:4166\cdot66\dot{6}=$ £3925·769662

15*s.* 4½*d.*

£3925 15*s.* 4½*d.* ans.

£1 = $\frac{1}{100}$	£3925·769662
·5 = $\frac{1}{200}$	39·25769662
	19·62884831
1 = $\frac{1}{100}$	3984·65620693 2d h. yr.'s prin.
·5 = $\frac{1}{200}$	39·84656206
	19·92328103
1 = $\frac{1}{100}$	4044·42605002 3d
·5 = $\frac{1}{200}$	40·44426050
	20·22213025
1 = $\frac{1}{100}$	4105·09244077 4th
·5 = $\frac{1}{200}$	41·05092440
	20·52546220
	£4166·66882737 proof.

Examples to be wrought, proved, and recited.

1. What is the value of a freehold estate, rented at $175 per annum, reckoning interest at $7 per cent?

2. What is the value of an English estate, let for £313 per annum; reckoning interest at £3 per cent?

3. What price may be asked for a house, in fee simple, let for $600 per annum, reckoning interest at $7½ per cent?

4. What is the value of a perpetual annuity of $810, reckoning interest at $6 per cent?

5. What is the amount of an annuity of $80, payable in half-yearly instalments, but forborne 3 years; reckoning annual interest at $6 per cent?

6. What would be the amount of a pension of $50, unpaid during 5 years, supposing government should allow interest upon it, at $5 per cent?

7. What is the amount of rent, payable half-yearly, on a house of $250 per annum, unpaid during 3 years, reckoning $6 per cent annual interest?

8. What is the present worth of an annuity of $300, payable by half-yearly instalments, to continue 3 years; interest reckoned at $7 per cent?

9. What is the present worth of four years' pension of $100, reckoning interest at $6 per cent?

10. What price might be asked for a rent charge of £55 per annum, payable in half-yearly instalments, to continue 4 years, reckoning annual interest at £4 per cent?

11. What is the value of a freehold house, rented at $300, payable in half-yearly instalments, to be possessed at the expiration of a three years' lease; reckoning interest at $5 per cent?

12. What sum should be paid down for an addition of 5 years to a lease of 5 years to come, in compensation for a rent of £319 per annum, reckoning interest at £4½ per cent?

13. Which is most advantageous, an immediate term of 15 years in an estate of £100 per annum, or the reversion of it in fee simple, after the expiration of 15 years, computing compound interest at £5 per cent?

The last example is from Webber, who gives for answer — *The 1st term, by* £75 18*s*. 7½*d*. The answer proves evidently, that his compound interest computed on one side was not set off by compound discount on the other; and it shows the absurdity of such modes of computation.

Exchange.

Principles of exchange.

What is exchange? — Exchange in commerce is the change of money in one place or country, for bills on some other place, or country; as a more convenient mode of discharging debts, than by cash remittances.

Has exchange any distinction of names? — One is called inland exchange; the other, foreign exchange.

What is the principal feature of foreign exchange? — Foreign exchange requires a comparison of the money of different cities and countries, and the proportioning of payments accordingly.

What are the grounds on which it proceeds? — In calculating exchange, the coin of a foreign country is supposed to contain that quantity of gold and silver of a determinate purity, which, by the regulations of its own mint, it ought to contain.

Is the same degree of purity to be found in the coin of all countries? — Civilized nations seem to be approximating to a

uniform standard, there is still therefore a diversity; British silver coin is of a higher degree of purity than that of the U. S.

Does this affect the exchange? — The exchange depends greatly on the intrinsic value, or purity, of the coin of the country in which payment is to be made.

What are the terms used in exchange? —

1. *Bills of exchange* are drafts on persons abroad, expressed in the money of their respective countries.

2. *The course of exchange* is the current price of bills between any two places, and is to be learned from the public prints; where this price varies, not in premium, but in the valuation of an exchange money itself, the course of exchange is the *variable price*.

3. *Par of exchange* is the real or reputed equal valuation, in the money of one country, of the money or bills of another.

4. *Intrinsic par* is the true valuation of the money of one country, from a comparison of it with the money of another, regarding only fineness and weight.

5. *Above par* and *below par* signify, the one, an estimation of foreign money or bills above their reputed foreign value; the other, a depreciation below it; the same terms are applied to the estimation of home, compared with foreign money, but contrariwise in the comparison.

6. *Premium* is the excess of price above par; *discount*, the deficiency from par.

7. *Currency* is the circulating medium of a country, including money and its substitutes, as bank bills; but sometimes, as in the W. I., it signifies money of account only, derived from an obsolete paper currency.

8. *Certain price* (where the term prevails) is the exchange unit, which, in exchange transactions between any two particular countries, is made the fixed basis of valuation; as the pound sterling between Britain and France; *variable price* is therefore exchange money of fluctuating exchange value, in reference to some opposite exchange unit.

Does the purity of coin alone affect the exchange? — The greater or smaller demand for bills of exchange, and the ease or difficulty of obtaining them, materially affects their value.

Is not exchange sometimes below par, when nominally above it? — Whenever the comparison of the money of a foreign with that of one's own country has been erroneously formed, the error will be corrected in time by a variation in the reputed par of exchange, which may thus be in a state the opposite to that seemingly indicated by it.

Have you an instance? — The British pound sterling has long been valued at $4·444; erroneously it is now considered, so much so, as to require a considerable rate of exchange, nominally favorable to Britain, to compensate the error.

What will occasion a reduction of the rate? — The little demand for foreign bills at any particular place or time, and a balance of trade unfavorable on the whole to a foreign country, will operate to reduce bills on it below par; that is, below the real value of the bill in the foreign country itself.

What is meant by the balance of trade? — The balance of trade is that difference between exports and imports which is payable in money.

When is it said to be favorable or the contrary? — When the amount of goods supplied to foreign nations is greater than their supplies to us, so that they become our debtors, the balance of trade is said to be in our favor; and as to the balance itself, undoubtedly is so.

In what manner does an unfavorable balance of trade reduce the bills of a nation below par? — An unfavorable balance creates a great demand for foreign bills in which to discharge it; the effect of this is, to give a premium to the foreign means of payment, which is the same with a depreciation of the home means.

Is this an unfavorable state of commerce? — Whether it be an unfavorable condition of things to a nation, depends on a variety of considerations; in private life, the man who supplies himself with goods, for which he pays cash, thinks himself in no state of adversity, if he can sell those goods to a profit; and imports exceeding exports are often reëxported so as to set off, or reduce, an unfavorable balance with one country, by a favorable balance with another.

Are there no means of making a double profit? — If goods can be bartered for goods, and the latter be sold for money, the merchant may perchance make a profit both ways; an equal barter, at all events, keeps down the rate of exchange, by preventing the accumulation of a balance.

What is the great principle of trade? — The great principle of trade is mutual advantage; when advantage obviously ceases on any one side, from whatever cause arising, the trade must cease.

What is the great principle producing a variation from the par of exchange? — The great and permanent cause of disparity of exchange, is the danger and expense of transporting the precious metals; it is affirmed, that where there is no depreciation of the currency, a premium beyond the real par

never exceeds the computed average cost of transporting specie.

What is the effect of an unfavorable state of exchange? — The effect of an unfavorable exchange, is to induce merchants indebted abroad to export goods; for if they can sell these without a loss, they save the premium on bills of exchange.

Is an unfavorable state of exchange always an effect of an unfavorable balance of trade? — *The balance of payments*, or debts to be presently liquidated, may render the exchange decidedly unfavorable for a time, though the balance of trade be as decidedly favorable.

How are the rates of exchange known? — The rates of exchange, and the values of foreign money, are to be found in tables, collected and printed for the use of merchants.

What is the unit of exchange? — *The exchange unit*, in any country, is the standard real or imaginary piece of money to which all computations of exchange between it and some certain other country are invariably conformed; as the pound sterling between Britain and France; the piastre between Britain and Spain; and is the same with *the certain price* of the London exchange.

When is it the same with the money unit? — Wherever prices are adjusted by a premium, there *the premium alone is the variable price;* and the money unit of either country becomes the exchange unit of a transaction, or otherwise, as it is, or is not, made the basis of valuation in that particular case.

Is there any mode of determining the absolute value of foreign compared with home money? — Coin of the same metal may be valued in any currency, by conjointly comparing their standards and weights, or comparing their weights of pure metal alone.

Can you exemplify it? — If the fine gold of the American half eagle weigh 123·134*gr.*, and that of the British sovereign weigh $113{\cdot}0011\dot{6}$*gr.*, by proportioning these numbers to $5, the value in silver of the half eagle, it will appear that the pound sterling is worth $4·588½.

$$\tfrac{113\cdot0011\dot{6}}{123\cdot134} \text{ of } \$5 = \$4\cdot5885.$$

What do you say of these weights and the standards? — The weights of pure gold mentioned may be found in books of authority, and the standards are the same.

By what rule are exchanges determined? — The arithmetical rules that govern exchange are single proportion direct and the reduction of composites.

Why do you say arithmetical? — The considerations, political and commercial, that influence exchange, are so various, that it is necessary to make such a distinction.

Premium and discount.

When does proportion arise in exchange? — Whenever money is to be valued at a premium or a discount, a question of proportion arises; because premium and discount are always estimated at a rate.

What step in the process is this? — Whenever moneys to be exchanged are not at par, the rate of exchange must first be found; for premium must be added to an exchange reducend, a discount must be deducted from it.

What is meant by exchange reducend? — To avoid circumlocution, *money to be estimated in an exchange value sought may be called the exchange reducend;* for it must be lowered or raised in denominated value, as the exchange unit is less or greater in value than its own.

Can you exemplify this? — If a sum in sterling money is to be valued in federal money, the sterling must be lowered to federal by reduction downward; for a dollar is of less value than a pound.

The contrary case? — If an amount of federal is to be valued in sterling money, the federal must be raised to sterling by reduction upward; for a pound is of greater value than a dollar.

How is premium proportioned in exchange? — Premium is proportioned to an exchange reducend, as 100 to the per-centage, or estimate given; for on every hundred of the money unit above par, the premium is allowed; as in the case of interest, on every hundred of the principal.

How is discount proportioned? — Discount is proportioned to an exchange reducend below par, as 100, plus the discount per cent, to the per-centage itself, or estimate given; for to every hundred of the money unit below par must be added the discount per cent, since nothing less will pass for a hundred in exchange.

Arbitration of Exchange.

In what other case of exchange does proportion arise? — Arbitration of exchange may require a proportional operation.

What is the arbitration of exchange? — Arbitration of exchange is a comparison of the exchange values in different countries, in order to determine whether the direct or indirect transmission of bills will be most advantageous.

What do you understand by indirect? — Indirect exchange is the transmission of bills, designed ultimately for one foreign country, through business channels of other foreign countries.

What distinctions prevail in this branch? — Indirect exchange is either between three places only, involving but a single arbitration of exchange; or it is between more than three, involves more than one arbitration, and may be denominated circuitous exchange.

Can you exemplify the advantage attending the indirect mode? — If a payment is to be made in Amsterdam, and the amount in federal money be of more value in sterling, than will discharge that debt in Amsterdam, it is evident that fewer dollars will be required by the indirect exchange through London, than by the direct exchange to Amsterdam.

Now what must be known, to secure this advantage? — To ascertain the possible advantage of an indirect exchange, either the several exchange values of different sums in the money of one of them must be known; or, one exchange value of home money being known, a second exchange value of the first exchange money, a third exchange value of the second exchange money, and so on, must be known, in order to proportion the reducend, through the known exchange value of home money, in all that are unknown.

Can you exemplify this? — To discover if any advantage will arise from remitting to Amsterdam by way of London, we must either know the value of British and Flemish money in different amounts of the federal unit, or an exchange value of federal money in the one, and an exchange value of that same one in money of the other.

Suppose the first case? — If two exchange values of different amounts in a third be known, either one can be proportioned to the same amount as the other; and thus the parity of value between all three will be found, each in money of its own country.

Can you exemplify it? — If the course of exchange be, London on Amsterdam, 33*s.* 9*d.* Flemish to the pound sterling; London on Paris, 32*d.* sterling to the ecu, 32*d.* sterling is then an article estimated in French money, from which we may find the exchange value of £1 in the same; the estimate found is $7\frac{1}{2}$ ecus; therefore $7\frac{1}{2}$ ecus, at the supposed existing rates of exchange, are on a par, in London, with 33*s.* 9*d.* Flemish.

$$\frac{240d.}{32d.} \text{ of 1 ecu}; = \frac{30}{4} = 7\tfrac{1}{2} \text{ ecus.}$$

How would you name and define a par thus found? — *The par of arbitration is the equivalent value of an exchange unit*

in moneys of different countries, at any contemporaneous rates of exchange.

How is the knowledge of this par a source of advantage? — The arbitrated par between any two foreign countries being known, if the direct exchange rise, while the indirect remains stationary, a drawer, who has it in his power, will take the indirect exchange.

Can you exemplify this? — Exchange between Paris and Amsterdam remaining at the former par of arbitration, of $7\frac{1}{2}$ ecus to 33*s.* 9*d.* Flemish, should the direct exchange between London and Amsterdam have risen to 40*s.* Flemish, a drawer on Amsterdam will prefer doing so by way of Paris; for thus he will save, additional charges excepted, 3*d.* Flemish on every pound sterling.

Should both exchanges rise? — One exchange must remain stationary, or a new par be arbitrated.

What were the other conditions of a question from which a par can be found? — The usual form in which a question of indirect or circuitous exchange presents itself is that in which one exchange value of home money is known, a second exchange value of the first exchange money, a third exchange value of the second exchange money; and so on, to the last exchange value sought.

Does any essential distinction exist between indirect and circuitous exchange? — No distinction whatever exists between indirect and circuitous exchange, but in the number of places comprehended in the latter term being more than three; they are governed by the same principle, and their results are obtained by the same rule; though commonly denominated, the former, simple arbitration; the latter, compound arbitration.

What is the rule of operation? — The rule of operation in every case of indirect or circuitous exchange, in which moneys of different countries are successively valued, one, in the money of another, is the reduction of composites; for the exchange reducend, to be valued in the money of a country, must be reduced to an expression of that value; and, successively, through intermediate exchange values, to the last value sought; as a pound is valued, by reduction through shillings and pence, in farthings.

What is the first step in such a reduction? — A series of moneys, estimated one in the other, with a view to reduction, should first be arranged like a table of composites.

What will be the order of the terms? — The table will begin on the right, with the reducend itself; it will be proceeded

in on the left below, with the unit, or other amount, in the denomination of the exchange reducend, made equal, on the immediate right, to the amount, known as its value, in some foreign money entering into the scheme of arbitration.

What will be the succession? — The succeeding terms form a series, in which the unit, or other amount of every preceding money is valued by the next succeeding; till we reach the ultimate exchange value sought, which, terminating the table on the lowest right, is thus connected with the reducend on the right above all.

Is the series uniform in its rise or descent? — For valuation in exchange, different moneys need not uniformly rise or descend in a table; it is only requisite, that a sum, in the denomination of a preceding money, be valued with precision in the next succeeding.

Can you exemplify such a table? — For an example of indirect exchange, let the price between London and Amsterdam be, 33*s*. 9*d*. Flemish to the pound sterling; between Amsterdam and Paris, 54*d*. Flemish to the ecu: arranging these terms from the reducend, we say, £1=33*s*. 9*d*. Flemish; 54*d*. Flemish=1 ecu; the exchange value in Paris being sought, the estimate found is, 7½ ecus.

Exchange reducend	= 1£
1£	=33*s*. 9*d*. Flem.
54*d*. Flemish	= 1 ecu.

Reduction.

33*s*. 9*d*.=405*d*.

$$\frac{9 : 405}{6 : \ 45}$$

7½ ecus. Ans.

May terms of the same money be in different denominations? — *Terms of the same money in an extemporaneous table of exchange, must be in the same denomination; or be reduced to the same before operating therewith.*

They are best reduced before insertion in the table.

How does the maxim apply to the first term? — The first term is found but once in the table, and must be in the same denomination with the exchange reducend.

How does the maxim apply to federal money? — Arithmetically speaking, there is but one denomination of federal money; namely, the dollar; its proportional values, as in all decimal computations, being expressed in the figuring.

Can you tell me what renders this sameness of denomination necessary? — Every reduction of composites is a proportional operation; for the reducend requires to be estimated in a different denomination, and this estimate must be sought from a unit, or other amount, in the same denomination with the reducend, in order that both articles may be of the same nature and name.

Can you exemplify this? — If £2 be the reducend, the article estimated is £1, or may be any sum in the denomination of pounds, the value whereof, in a different denomination, is expressed in the estimate given. $\frac{£2}{£1}$ of $20s. = 40s.$

Why should it be necessary in the intervening terms? — An exchange reducend is reduced to the value of every money expressed in its table; and in order to this, the unit, or other amount, of every preceding money becomes an article estimated in the value of that next succeeding; but the article requiring estimate is in the same preceding money, and both articles must be in the same denomination.

Can you exemplify this also? — We see it in the example lately given; £1 is there reduced to 33*s.* 9*d.* Flemish; this money is to be reduced to French money, by making 54*d.* Flemish the article estimated at 1 ecu; therefore the article requiring estimate must be reduced from Flemish shillings to Flemish pence. $\frac{33s.\ 9d.}{54d.} \times 1$ ecu; $= \frac{405}{54}$.

What are the reductions applied? — The reductions applied in a single operation of circuitous exchange may be of both kinds; for a preceding money may present a greater or a smaller number than the money succeeding; if it be greater, the value is less, and the reduction to the succeeding money is upward, that number may be diminished in proportion to the increase of denominated value.

When will the reduction be uniform? — When the terms on the left of the table are units only, the reduction will be uniformly downward; for every succeeding money from the first, is then expressed in larger numbers than the money next preceding; the reduction is therefore as from a higher to a lower denomination; and when the terms on the *right* of the table are units only, the reduction is as from a lower to a higher denomination.

How may these reductions most conveniently be performed? — Reduction upward being division, will most conveniently be performed on the final product obtained by reduction downward; for thus intermediate fractions will be avoided.

Will the result be the same? — If the product of the exchange reducend and of all the terms on the right of the table, multiplied one into another, be divided by the product of all the terms on the left of the table, the ultimate exchange value given will be the same, as if any reduction upward were made in the order of its occurrence.

How does this appear? — The terms in reduction upward are, each, divisors of factors in the reduction downward; their product, consequently, may be a multiple divisor of the final product in the reduction downward.

Cannot this double process of reduction be abridged? — The numbers on the left of the table are divisors, on the right they are multipliers; therefore if any two numbers, one on each side, are equal, they may both be omitted in the reduction. Common measures may also be applied to both sides of the table, by reducing a term on each side, or several terms for one, by submultiples.

When is this best done? — For security's sake, contractions should be used only in the proof.

What is the issue of this reduction? — The result of every reduction through one or more foreign moneys to an ultimate exchange value is the par of arbitration in the ultimate value; and from the same table the arbitrated par of every other money entering into it may be found, by considering it as an ultimate value, and operating in the reduction accordingly.

Does no question of time arise in the calculation of exchange? — Time is an important element in bills of exchange, but is considered in the premium, or in the value of the bill, without affecting the process of calculating the exchange itself.

What tables are essential to the practice of exchange? — The business of exchange cannot be satisfactorily conducted without copious and authoritative tables of the moneys of the principal seats of commerce, and their value in the money of one's own country.

How is brokerage regulated in respect to exchange? — Brokerage is charged on the par amount of a transaction, in reduction of the proceeds.

Can you now recite the maxims and rule?

MAXIMS IN EXCHANGE.

Premium is proportioned to an exchange reducend, as 100 to the per-centage, or estimate given; and to the reducend the premium found must be added before reduction.

Discount is proportioned to an exchange reducend, as 100, plus the discount per cent, to the per-centage itself, or estimate given; and from the reducend the discount found must be subtracted before the reduction.

Proof, as in single proportion direct.

The direct exchange value of a reducend to a lower exchange unit, is the product of its amount, plus the premium, or minus the discount, if any, multiplied into the par of its unit, valued in the exchange money sought; and of a reducend to a higher exchange unit, is the quotient, in the same amount, of the par of the exchange unit, valued in money of the reducend itself.

Proof of one exchange by the opposite.

RULE IN THE ARBITRATION OF EXCHANGE,
(INDIRECT AND CIRCUITOUS.)

To find the par of arbitration between any two or more places, arrange the moneys that enter into the scheme as a table of composites, beginning, on the right, with the exchange reducend; proceeding, on the left below, with a sum in the denomination of the reducend, to be valued on the immediate right in the exchange money apportioned to it in the question. In this manner continue the series, from the left to the right, by valuing a sum in the denomination of every preceding money, as it occurs, in that money, yet unvalued, which is apportioned thereto, till the ultimate exchange money complete the table on the right. The values on the left are terms of reduction upward; those on the right, of reduction downward.

Multiply the exchange reducend into every factor on the right of the table; if there be units only on the left, the product is the answer; if there be more or less than units on the left, divide the entire reducend product, by the product of all the factors on the left; the quotient is the answer.

Common fractions in a term may be obliterated in the usual manner, extending the process to the corresponding value on the other side of the table.

Any pair of equal numbers, namely, a single number on each side of the table, may be omitted in the reduction.

When two foreign values of different amounts of the home money are known, the par of arbitration is found, by proportioning the same amount of home money to both exchange values.

Proof. Reverse the operation; reducing any factors, one on each side of the table, to their least terms, by the same common measure; or a single factor on one side, and two or

more factors on the other, by submultiples of the same common measure.

APPLICATION.

1. What is the amount in federal money, of £157 sterling, at the reputed par value of $\$4{\cdot}44\dot{4}$ to the pound; exchange on London, at 30 days, being at £$7\frac{7}{8}$ per cent premium?

e s is the ratio,

$\frac{157}{100}$ of £$7\frac{7}{8}$, prem. per cent; =157

$$
\begin{array}{r}
157 \\
\cdot 07\frac{7}{8} \\
\hline
8:1099 \\
137375 \\
\hline
\text{Premium } £12{\cdot}36375
\end{array}
$$

$$
\begin{array}{rl}
£157 & \text{bill of exchange.} \\
12{\cdot}36375 & \text{premium added.} \\
\hline
169{\cdot}36375 & \\
4\frac{4}{9} & =\$4{\cdot}44\dot{4} \text{ par value of } £1. \\
\hline
9:677{\cdot}45500 & \\
75{\cdot}272\dot{7} & \\
\hline
\$752{\cdot}727\dot{7} & \text{answer.}
\end{array}
$$

2. What is the value, in sterling money, of $\$752{\cdot}72\dot{7}$, computing federal money at $\$4{\cdot}44\dot{4}$ to the pound, and the nominal par at $\$7\frac{7}{8}$ per cent discount?

e s is the ratio,

$\frac{752{\cdot}72\dot{7}}{107{\cdot}875}$ of $\$7\frac{7}{8}$, discount per cent; $=752{\cdot}72\dot{7}$

$$
\begin{array}{r}
752{\cdot}72\dot{7} \\
7\frac{7}{8} \\
\hline
8:5269{\cdot}094\dot{4} \\
658{\cdot}6368 \\
\hline
107{\cdot}875:5927{\cdot}7312=54{\cdot}9499 \text{ disc.}
\end{array}
$$

$$\begin{array}{r l}
\$752{\cdot}72\dot{7} & \text{exchange reducend.} \\
54{\cdot}949 & \text{discount deducted.}
\end{array}$$

$$\begin{array}{r l}
4\tfrac{4}{9} : 697{\cdot}77\dot{7} & \\
9 \quad\quad 9 & \\
\hline
4{,}0 : 627{,}9{\cdot}999 = £156{\cdot}9999 & \text{answer.}
\end{array}$$

These two examples prove each other.

3. What are the different exchange values, in federal money, of £157 sterling, at the reputed par, and a premium of £8 per cent; and at the American statute value of \$4·80 to the pound?

$\frac{157}{100}$ of £8, prem.$=157\times{\cdot}08=12{\cdot}56$ premium.

$$\begin{array}{r l}
£157 & \text{ex. reducend.} \\
12\times{\cdot}4=\$4{\cdot}8, & \text{statute par.} \\
\hline
1884 & \\
{\cdot}4 & \\
\hline
\$753{\cdot}6 & \text{answer.}
\end{array}
\qquad
\begin{array}{r l}
£157 & \text{ex. reducend.} \\
12{\cdot}56 & \text{premium added.} \\
\hline
169{\cdot}56 & \\
4\tfrac{4}{9}\$, & \text{par of } £1 \\
\hline
9 : 678{\cdot}24 & \\
75{\cdot}36 & \\
\hline
\$753{\cdot}60 & \text{answer.}
\end{array}$$

Hence it appears that the statute value of \$4·80 to the pound sterling, is equivalent precisely to a computation of the pound at \$4·44$\dot{4}$, and a premium of £8 per cent.

4. A merchant in Britain draws on Amsterdam for £782 sterling; how many pounds Flemish will that amount to, exchange being at 34*s.* 8*d.* to the pound sterling?

$$\begin{array}{r l}
£782 & \text{exchange reducend.} \\
34\tfrac{2}{3}s.=34s.\ 8d. & \text{par of } £1. \\
\hline
3 : 1564 & \\
\hline
521{\cdot}\dot{3} & \\
3128 & \\
2346 & \\
\hline
2{,}0 : 2710{,}9{\cdot}\dot{3} & \\
\hline
£1355\ 9s.\ 4d. & \text{ans. in pounds [Flemish.}
\end{array}$$

$$\begin{array}{r l}
34\tfrac{2}{3} : 27109{\cdot}\dot{3}s. & \\
3 \quad\quad\quad 3 & \\
\hline
104 : 81328 = £782 & \text{proof.}
\end{array}$$

5. *Indirect exchange.* London on Paris, 25 francs to the pound sterling. London on Cadiz, 40*d.* to the piastre; what is the arbitrated price?

$\frac{240d.}{40d.}$ of 1 piastre; $=6$ piastres. Ans.

This is evidently the par of the British exchange unit, as between Paris and Cadiz; for, at London supposed prices, 6 piastres are an exchange value equivalent to 25 francs. But if, from these terms, the arbitrated price of the two foreign moneys be sought with reference to the exchange unit of one of them, say the doubloon of 4 piastres, the operation must be resumed, and will be as follows.

$\frac{4p.}{6p.}$ of 25 francs; $=16{\cdot}\dot{6}\dot{6}$ francs $=1$ doubloon. Ans.

Compare with Kelly, vol. ii. pp. 118, 119.

6. Circuitous exchange of £1000 on Cadiz. Rates: London on Amsterdam, 35*s.* Flemish to the pound sterling; Amsterdam on Paris, 58*d.* Flemish to the ecu; Paris on Venice, 100 ecus for 60 ducats; Venice on Cadiz, 360 mervadies to the ducat; how many piastres, of 272 mervadies, will the thousand pounds amount to in Cadiz?

35*s.* F. $\times$ 12 $=$ 420*d.* Flemish.

			Least terms.
Exchange reducend	$=$ 1000£		10
1£	$=$ 420*d.* Flemish	1	210
58*d.* Flemish	$=$ 1 ecu	29	1
100 ecus	$=$ 60 Venetian ducats	1	30
1 ducat	$=$ 360 mervadies	1	45
272 mervadies	$=$ 1 piastre	17	1

Common Measures.

100, of £1000 and 100 ecus; 2, of 420*d.* and 58*d.* Flemish; 8, of 360 and 272 mervadies; 2, of 34 mervadies obtained from the preceding reduction, and 60 ducats.

Reduction at large.

272	420000
5800	60
2176	25200000
1360	60 $\times 6=360$
1577600	1512000000
	6

15776,00 : 90720000,00 $=5750\frac{1000}{1972}$ piastres. Ans.

Reduction by least terms.

29	2100
17	30
203	63000
29	5 $\times 9 = 45$
493	315000
	9

$493 : 2835000 = 5750\frac{250}{493}$ piastres. Ans.

Reverse operation.

				Least terms.
Exchange reducend	=	$5750\frac{1000}{1972}$ piastres		$575\frac{100}{1972}$
1 piastre	=	272 mervadies	1	34
360 mervadies	=	1 Venetian ducat	9	1
60 ducats	=	100 ecus	3	1
1 ecu	=	58*d.* Flemish	1	29
420*d.* Flemish	=	1£	21	1

Common Measures.

Three numbers on the left are divided by 10; on the right, two numbers; viz. the reducend, once, and 100 ecus, twice.

8, of 272 mervadies; 4, of 36 mervadies; 2, of 6 ducats; 2, of 58*d.* and 42*d.*

21	34
9	29
189	306
3	68
567	986
	$575\frac{100}{1972}$
	50
	4930
	6902
	4930

$567 : 567000 = £1000$ proof.

The terms of an indirect exchange, similarly conditioned, are arranged and operated with precisely in the same manner.

This example, and elegant mode of displaying the reduced terms, *are copied*, with considerable alterations of expression and arrange-

ment, from the Encyclopædia Britannica, 3d ed. Whoever will be at the pains of working the same example by the two modes, will probably assent to the old adage, that the longest way about is often the shortest way home. Before consulting that work, the author had formed his method and rule of indirect and circuitous exchange; the encyclopædist adopts substantially the same plan, though with a formidable array of linked ratios: of these we get rid, and hear nothing of terms of demand, antecedents and consequents, *odd* terms and *chain* rules, that can never chain themselves to the mind, for the very reason that they are odd. It is not a little remarkable, that in none of the printed rules is any caution given about sameness of denomination. Indeed, from the manner in which one is accustomed to see this subject treated, it may well be reckoned among the veriest of arithmetical riddles; a solution of that riddle is here attempted in behalf of the novice; with what success is for the initiated to say. Elucidation of fundamental principles, and concise exemplification of the practice, is all that can be given to purpose in a work on arithmetic. Like book-keeping, and much more so, exchange must have its distinct treatise. Such a treatise is to be found in Kelly's Universal Cambist, a work of great price, but prepared at vast expense, and seemingly indispensable to the great merchant and broker; in whose offices only can the wide field covered by exchange be accurately traced.

Examples to be wrought, proved, and recited.

1. Exchange on London, valued, in federal money, at \$5610·17⅛, at \$4·59 to the pound; what is the amount in sterling?

2. Exchange on London for £368 19*s*. 10½*d*., at \$4·8 to the pound sterling; what is the value in federal money?

3. Exchange on London, valued, in federal money, at \$785·716, computing $\$4{\cdot}44\dot{4}$ to the pound, and a nominal discount of \$9½ per cent; what is the sterling?

4. Exchange on London for £200, at $\$4{\cdot}44\dot{4}$ to the pound sterling, and a nominal premium of £8⅜ per cent; what is the federal money?

5. Exchange on Paris, for \$286·216, at \$·18¾ to the franc; what is the exchange value?

6. Exchange on Paris, for 163 francs, 13 centimes (163·13*fr.*) at \$·18¾ to the franc; what is the exchange value?

7. Exchange: London on Cadiz, for 2163 piastres, 4 reals; at 38⅜*d*. sterling to the piastre (or piece of 8 reals), what is the sterling value?

8. Exchange (indirect): London on Amsterdam, at 405*d*. Flemish to the pound sterling; London on Paris, 32*d*. to the ecu; if the par of arbitration between Amsterdam and Paris

rise to 56*d*. Flemish per ecu, what will be the advantage per cent, by drawing at London on Paris, through Amsterdam?

(*From Encycl. Brit. Answer*, £3 11*s*. $5\frac{1}{7}$*d*.)

9. Exchange (circuitous): London on Cadiz, for £1000. Rates: London on Amsterdam, 10 florins, 10 stivers, (10·5*fl*.), to the pound sterling. Amsterdam on Paris, 60 pence Flemish, for the ecu of 3 francs. Paris on Cadiz, 15 francs for 1 doubloon of 4 dollars of exchange; what is the Cadiz exchange value? (*From Kelly. Answer*, 5600 *dollars*.)

10. Exchange (circuitous): Amsterdam on London, for £400 Flemish. Rates: Amsterdam on Paris, 56*d*. Flemish to the ecu. Paris on Venice, 100 ecus to 60 ducats. Venice on Hamburgh, a ducat to 100*d*. Flemish. Hamburgh on Lisbon, 50*d*. Flemish per crusade of 400 rees. Lisbon on London, the millree to 64*d*. sterling. What is the sterling amount?

(*From Encycl. Brit. Answer*, £219 8*s*. $6\frac{4}{7}$*d*.)

11. Exchange: Amsterdam on London, for £400 Flemish. At 36*s*. 10*d*. to the pound sterling, what is the result of a comparison between the direct exchange and the circuitous described in No. 9?

12. If the exchange value of £1 be, in Amsterdam, 35*s*. Flemish; if 3*s*. 6*d*. Flemish be, in Lisbon, 400 rees; if 480 rees be, in Paris, 3 francs; what is the exchange value of £1 at Paris?

Compensation.

What is alligation? — *Alligation* is from a Latin word, signifying *to tie together;* it is made the name of a rule in arithmetic, because it has been usual, in solving certain questions under that rule, to extend lines from one number to another.

What is the subject of the rule? — Alligation treats of proportional commixtion and compensatory adjustments.

What is meant by the latter? — Compensatory adjustments include the philosophical uses made of this rule, for discovering a mean, and proportioning quantities to a mean; and since every application of the rule implies compensation of a greater by a smaller, and the reverse, perhaps we may be allowed to give it the name of compensation.

By what name is commixtion best known? — The mixture of things of different values for the sake of an increased gain, is well known by the name of adulteration.

Does the rule then teach to adulterate? — The rule does *not* teach to adulterate; but to deduce from known mixtures, a true proportional price; and from known prices, a true proportional mixture.

Does it serve any purpose contrary to the arts of adulteration? — In regard to metals, we derive from the rule a mode of detecting adulteration.

COMPENSATION MEDIAL.

What is a mean? — A mean is something intermediate; in arithmetic, it is a number intermediate between a greater and a less.

How is such a number found? — Compensation medial is that branch of the rule of proportion direct by which we find a mean or intermediate number; as between higher and lower rates, between superior and inferior qualities blended together.

Where is the compensation? — The compensation is in the mean rate, which is below the higher, and above the lower rate.

How may we value mixtures? — The entire value of a mixture is the sum total of the products of every ingredient, multiplied into its price, or particular rate at which it is estimated.

What would a resulting rate be? — A rate obtained from the entire value will be a mean, or intermediate, rate; for, by the mixture, higher and lower priced articles are commingled; their prices, consequently, represented by number, are commingled in the entire value.

How is the rate found? — A mean rate is found by division of the entire value by the sum of the quantities mingled; for thus we obtain the mean value of a unit of the quantities.

Can you exemplify it? — If 2 gallons of two dollar wine be mingled with 3 gallons of wine at three dollars, the mean price, or rate, of the mixed gallon is \$2·60; for \$2·60 is a fifth part of \$13.

Rule.

$$\begin{array}{lcl} \$2\times 2\,ga. & = & \$4 \\ 3\times 3 & = & 9 \\ \text{—} & & \text{—} \\ 5 & : & 13 = \$2{\cdot}6 \text{ ans.} \end{array}$$

How do you bring this process within the proportional form? — In the example three things are expressed or obviously implied; the ingredients or article estimated; namely, 2 gallons and 3 gallons, estimated respectively at \$2 and \$3, which constitute the rate given; and implied in the sum of the quantities, or article requiring estimate.

Collective ratio.

$$\frac{\overline{2\times 2}+\overline{3\times 3}}{5} = \$2{\cdot}6.$$

What is *e s*? — The estimate sought is a mean rate; and the proportional form is that of a collective ratio, in which the parts of the estimate given, as in equation of payments, are multiplied, each into its proper article; for they could not truly be represented in a single sum.

What is the proof? — In compensation medial, the product of the sum of the quantities, multiplied into the mean rate, must equal the sum of the products of each particular quantity, multiplied into its particular rate; for thus gain and loss are compensated.

COMPENSATION ALTERNATE.

What is compensation alternate? — Compensation alternate is a proportional method, by which higher and lower rates are made to compensate each other, in an alternating process from a higher to a lower rate.

In what does the distinction of the two compensations essentially consist? — Compensation medial proportions a mean to quantities; compensation alternate proportions quantities to a mean.

To proportionate quantities to a mean rate specified.

How may we adjust quantities to be mingled, proportionally to a mean value specified? — In a mixture of intermediate value, articles of a greater price are rated at a smaller, and compensated, by rating others of a lower, at a higher, price; quantities therefore are to be so proportioned, that there shall be a sufficiency of the cheaper articles, to counterbalance, by the excess of the mean rate above their particular rate, the loss that would otherwise accrue on the higher priced articles; and so of superior and inferior qualities.

What will be the adjustment when the excess in value of one article is equal to the deficiency in another? — When one ingredient exceeds in value the mean value specified, so as exactly to counterbalance the deficiency in value of another, the mixture will contain equal quantities of each; for thus the gain and loss are exactly equal.

What, when excess and deficiency are not equal? — Excess and deficiency of value unequally proportionate to the mean value require a compensatory adjustment of the ingredients.

How is the compensation made? — Since the loss on higher values is to be compensated by the gain on lower, the proportion of ingredients in a mixture will vary, not each one with its own difference of value, but each one of value greater than the mean with the difference of a less, and each one of *less value* with the difference of a greater.

Wherefore? — Because the larger the difference of a value from the mean value, the larger quantity must there be of some ingredient on the opposite side of the mean, to equalize, by a compensative deficiency, or excess of value, the rate on the whole.

Then what will be the proportions? — Ingredients therefore of a higher value will be proportioned in a mixture directly as the difference from the mean of a lower value; and these will be alike proportioned as the difference from the mean of some higher value; thus the differences are alternated, or interchanged.

Does the alternation apply to any number of values? — It matters not what the number of ingredients may be, provided there are both greater and less than the mean in value; for the excess of the greater may be placed on one or more of the less, and the contrary.

Can you exemplify the case of an equal excess and deficiency? — If articles of ten and twelve dollars price are to be mingled, and sold at the rate of eleven, since each differs alike from the mean, there must be an equal quantity of each; for the ten dollar ingredient will thus be increased to eleven, and the twelve dollar reduced to eleven.

Can you exemplify the case of inequality? — If an eleven dollar mixture is sought to be made up of ingredients, some rated at ten, others at thirteen, dollars, for the same weight, number, &c., the difference from the mean in value being, on one side, 2, on the other side, 1, the quantities will differ as 1 and 2.

How are the differences interchanged? — The ten dollar ingredient is a unit short of the mean in value; its elevation to the mean is therefore more than compensated by one quantity of the thirteen dollar ingredient being reduced, in price, to the mean; the thirteen dollar ingredient differs twice as much in excess of the mean value, as the other falls below it; and therefore exactly compensates a double quantity of the ten dollar ingredient, raised, by the mixture, to the price of the mean.

What is the notation? — If the extreme rates be placed in column on the left, having the mean in the centre, on the right, the interchanged differences may conveniently be noted on the farther right, for multiplication into their respective alternated values.

Rule.

$$\begin{array}{l} \\ \$13 \\ \\ \$10 \end{array}\left.\vphantom{\begin{array}{l}1\\1\\1\\1\end{array}}\right\}\begin{array}{c}\text{mean,}\\ \$11\end{array}\left\{\begin{array}{l}\text{ans.}\\ 1\times13=13\\ \\ 2\times10=20\end{array}\right.$$

$$\begin{array}{ccc} \overline{} & & \overline{} \\ 3 & : & 33=\$11 \end{array}$$

[proof.

Ratios formed.

$\frac{1}{3}+\frac{2}{3}=1$ mixture in the proportions required.

What will the answer consist of? — The question being, what proportion, or proportional quantities, at the extreme rates, will bear the mean rate specified; it is answered by the differences of quantity found and interchanged; for these compensate one another in value, and the values are derived from the quantities.

What then do the differences represent? — The differences represent proportional quantities; or, simply, proportions, if considered in the nature of turns.

What is the proportional form? — The proportional form is, of the sum of the differences to each separately, or of each one in the whole; for the mixture consists of all the proportions or parts; the differences indicate the parts, and their sum is the whole.

Can you exemplify it? — In the last example, the thirteen dollar ingredient constitutes $\frac{1}{3}$ of the mixture; the ten dollar ingredient constitutes $\frac{2}{3}$; the two proportionate quantities making $\frac{3}{3}$, or 1 mixture in the proportions required.

Does the process fall under any rule of proportion? — Compensation alternate involves no general rule of proportion, but the definition only, and its own peculiar rule, in the process of alternation; the purpose of this being the formation of unequal articles, not the determining of their estimates.

How is the operation proved? — If the differences found be multiplied respectively into their rates assigned by alternation, the sum of their products will be the value of a mixture entire, composed of the quantities expressed by the differences.

What is the inference? — If this entire value be measured by the sum of the differences, or parts, at the mean rate specified, the ingredients have been justly proportioned, and the operation is right.

Specific gravity.

Which is the heaviest of metals? — Gold is heavier than any other metal, except plátina, a metal of rare occurrence.

To what is the property of greater weight owing? — Greater weight is owing to greater density, or material in greater quantity under the same bulk.

What is difference of density called? — The density of a body determined from that of some other body considered as the standard, is called its specific gravity.

Are there any means of ascertaining the specific gravity of bodies? — The specific gravity of one body, as compared with that of another of the same weight, is inversely as their comparative bulks; for the greater the bulk, the less is the density.

Have you an instance? — Wood and metal of equal weight differ vastly in their bulk, and consequently in their density.

Does this afford any means of detecting the adulteration of metals? — Bodies displace water in proportion to their bulk; the same weight therefore of an adulterate and of a pure metal will displace different weights of water; and the adulterate will be of a specific gravity intermediate between equal weights of the alloy alone, and of the metal which is counterfeited.

What are the specific gravities of the moneyed metals, and how determined?

A cubic foot of pure water weighs 1000*oz.*, is the standard; and its specific gravity is called 1.

———————	gold of 24*ca.* weighs,	19258*oz.*; sp. grav.,	19·258
———————	silver, cast, weighs,	10474*oz.* ———	10·474
———————	copper, cast, weighs,	8788*oz.* ———	8·788

The gold of Dutch ducats has been without alloy, and above the standard here given. The specific gravity of plátina is 19·5.

To proportionate ingredients to a mean rate and total quantity specified.

How are quantities adjusted to a specified mean value, and to a limitation of entire quantity? — When a mixture is limited, both in totality of quantity, and in rate, the ingredients are first to be proportioned to the mean rate specified; hence we obtain the proportion of all the quantities to every particular quantity; and thence, the ratio of every ingredient separately in the totality specified; for in every mass of a mixture, the proportions must be the same, or the true rate will differ from the mean specified.

Can you exemplify the case? — If the ingredients of a mixture be 3℔ of one article, and 4℔ of another, their respective ratios are $\frac{3}{7}$ and $\frac{4}{7}$; increase the whole quantity to 30℔, and of one ingredient there will be $\frac{3}{7}$ of 30℔; of the other, $\frac{4}{7}$ of 30℔. $\frac{3}{7}\times 30=\frac{90}{7}$. $\frac{4}{7}\times 30=\frac{120}{7}$. $\frac{90+120}{7}=30$, proof.

What will be the proof? — If the work be correct, the sum of the parts formed will be equal to the totality specified.

To proportionate ingredients to a mean rate, and limitation of an ingredient specified.

How are quantities adjusted to a specified mean rate, and to the limitation of an ingredient? — When a mixture is limited in rate, and to a quantity specified of one ingredient, after proportioning the quantities of every ingredient, without exception, to the mean given, we obtain one proportion therein of the limited ingredient to the unlimited; and as the latter

will vary with the former, the proportion of each unlimited to the limited ingredient is also its ratio of the limited quantity specified.

Can you exemplify the case? — If in an eleven dollar mixture, a ten dollar ingredient enter to the extent certain of 10℔; after finding 2℔ as its quantity proportional to 1℔ of an unlimited ingredient, rated at $13, we shall then have the ratio ½ of 10℔, for the proportionally increased quantity of the thirteen dollar article; since thus the proportion of the ingredients to each other is still as 2 to 1.

$$\left.\begin{array}{l}\$13\\ \\ \$10\end{array}\right\}\begin{array}{c}\text{mean,}\\ \$11\\ \end{array}\left\{\begin{array}{lr}\text{differences} & 1\\ \text{alter-} & \\ \text{nated} & 2\end{array}\right.$$

$$\begin{array}{rl}\tfrac{1}{2}\times 10℔ = 5℔ \text{ at } \$13 = & \$65\\ \text{add } 10℔ \text{ at } \$10 = & 100\\ \hline 15 \quad : & 165 = \$11,\\ & \text{[proof.}\end{array}$$

What will be the proof? — If the limited quantity specified, added to the proportionate quantities found, measure the entire value by the mean rate specified, the work is correct.

Should there be more limited ingredients than one, what would be the proceeding? — Of two ingredients, the quantity of one only must be limited, or there can be no proportioning to a mean; and one quantity limited must not be affected by the limitation of another; or one of the two limitations will fail.

What is the inference? — Therefore, when more than a single ingredient is limited, the operation from the beginning, for each, is best conducted apart, lest the differences interchanged be compared erroneously.

In what respect? — Erroneously, should the limited articles be compared one with another; but the quantities found by separate operations, being all proportioned to the same mean, may then be added together, and proved.

Has water any moneyed value, compared with other liquors? — Water is of so small a pecuniary value, that mingled with other liquors, it is reckoned as nothing.

How are alloys considered as to value? — Alloys are considered as of no value, in comparison with the precious metals they qualify or adulterate.

Do you recollect what a cáract was said to be? — A cáract *is one twenty*-fourth part of any weight; used ordinarily of *small weights* of the precious metals.

How is the denomination applied? — The minter's ounce of gold is divided into 24 cáracts; fine gold consists of these 24 parts in a state of purity; but if we suppose the gold to be alloyed, the alloy to be separated, and to equal one or more of such parts, the gold is then said to be of as many cáracts fine as are equal to 24, less the alloy.

Can you now recite the maxims?

MAXIMS IN COMPENSATION.

Medial.

A mean rate sought is as the sum of the quantities in a mixture to their entire value.

Proof. The product of the sum of the quantities, multiplied into the mean rate, equals the sum of the products of each particular quantity, multiplied into its particular rate.

Alternate.

Ingredients are proportioned to a mean rate specified, by alternation; those, namely, of a higher value, as the sum of the differences from the mean to particular differences of lower value; ingredients of a lower value, as the same sum to particular differences of higher value.

Proof, the sum of the differences measuring the entire value by the mean specified.

Ingredients proportioned to a mean rate by alternation enter, in the same proportion, into a total quantity specified.

Proof, the sum of the portions found being equal to the totality given.

Ingredients proportioned, without exception, to a mean rate by alternation, enter into a particular quantity specified, in the ratio found of the limited to each unlimited quantity in the mean.

Proof, the sum of the quantities limited and found, measuring the entire value by the mean rate.

N. B. For more than one limited ingredient, let there be as many distinct operations; if correct, the results, added together, will measure their entire value by the mean.

APPLICATION.

1. A silversmith makes a mass of silver, consisting, 12℔ of 6*oz.* fine in the ℔, 8℔ of 7*oz.* fine, 10℔ of 8*oz.* fine; what is the fineness of every ℔ of the mass?

Collective ratio,				values.
$\frac{6\times12+7\times8+8\times10}{30}$	rate	6*oz.*	$\times$ 12℔	$=$72*oz.* fine.
	of	7	$\times$ 8	$=$56
	ingred.	8	$\times$ 10	$=$80

Sum of the quantities 3,0 : 20,8=6*oz.* 18*dw.* 16*gr.*, [ans.

6*oz.* 18*dw.* 16*gr.*
$3\times10=30$

1	8	16	0
		10	
17℔	4*oz.*	0*dw.*	
6	0	0	
3	4	0	
3	4	0	
℔30	0	0 proof.	

Proof of the present example might long puzzle a beginner. The weights added to the product by composite multiplication, are the proportions of alloy, omitted, of course, in the estimation of fineness.

2. A grocer mixed teas of different qualities, 5℔ at 7*s.* per ℔, 9℔ at 8*s.* 6*d.*, and 14½℔ at 5*s.* 10*d.*; what was the mean rate of the mixture?

rate	7*s.*	$\times$ 5℔	$=$35*s.*
of	8·5	$\times$ 9	$=$76·5
ingred.	$5\frac{5}{6}$	$\times$ 14·5	$=84{\cdot}58\dot{3}$

Sum of the quantities, 28·5 : $196{\cdot}08\dot{3}$=6·880116*s.*
12*d.*=1*s.*

10·561392

Ans., 6*s.* 10¾*d.*, mean rate by the ℔. 4*f.*=1*d.*

2·245568

$4d.=\frac{1}{3}s.$	28·5 lb
	6·5$s.$=6$s.$ 6$d.$
	1425
	1710
	185·25
$\frac{1}{2}d.=\frac{1}{8}$ of $\frac{1}{3}$	9·5
$\frac{1}{4}d.=\frac{1}{2}$ of $\frac{1}{24}$	1·1875
	·59375
	196·53125$s.$ proof.

The remaining fraction must be taken as an additional farthing, or the vendor would lose.

3. Gold of 17, 18, 22, and 24 cáracts fine, is to be melted together, to form a mass of 21 cáracts fine; what must be the proportions?

				Answer.	Values.
rate	24$ca.$	mean,	differ.	4 of 24$ca.$	=96
	22			3 of 22	=66
of		21$ca.$	alter-		
	17			3 of 17	=51
ingred.	18		nated.	1 of 18	=18
			Sum of the differ.	11 :	231=21$ca.$ proof.

Ratios formed.

$\frac{4}{11}+\frac{3}{11}+\frac{3}{11}+\frac{1}{11}=1$ mass of 11 parts, and 4 qualities.

4. Of currants at 4$d.$, 6$d.$, and 9$d.$, per lb, what must be the proportions intermingled to make a cwt. English, priced at 8$d.$ per lb?

rate	9$d.$	mean,	differ.	$\overline{4+2}\times 9=54d.$
of	6$d.$	8$d.$	alter-	$1\times 6=$ 6
ingred.	4$d.$		nated.	$1\times 4=$ 4
			Sum 8 :	64=8$d.$ proof of alter-[nation.

Ratios.

$\frac{6}{8}$ of 112 lb $=\frac{672}{8}$ =84 lb at 9$d.$ }
$\frac{1}{8}$ of 112 lb =14 at 6$d.$ } Answer.
$\frac{1}{8}$ of 112 lb =14 at 4$d.$ }

112 lb totality. Proof.

5. Suppose the adulterate crown made for Hiero to have weighed 10℔, and to have displaced 64℔ of water; a mass of pure gold of the same weight to displace 52℔; a mass of copper, also of 10℔, to displace 92℔ of water; what was the alloy, and what the pure gold, in the crown?

				Parts.
rate of 10℔ of copper,	92℔	adulte-	differ.	$12 \times 92 = 1104$℔
		64℔	alter-	
rate of 10℔ of gold,	52℔	rate	nated	$28 \times 52 = 1456$
			Sum, 4,0 :	256,0 = 64℔, [proof of alternation.

Ratios.			
copper $\frac{12}{40}$ of 10℔	adulterate crown.	$=\frac{12}{4}=3$℔ of alloy	Ans.
gold $\frac{28}{40}$ of 10℔		$=\frac{28}{4}=7$℔ of the precious [metal.	
		10℔ totality. Proof.	

The example is taken from Hutton, and the quantities, it is imagined, are given only as a proportional illustration; for who would think of putting a weight of 10℔ on his head, by way of ornament? and how should that weight displace 64℔ of water? If the proportions however be correct, this suffices to illustrate the means of detecting a fraud which seems to set human ingenuity at defiance; means, the discovery of which might well carry Archimédes from the bath to his home, as the story is told, exclaiming, *Εὕρηκα!* "I have found it!"

The rationale is probably as follows. The water displaced by the adulterate crown is a quantity intermediate between the quantities displaced by equal weights of the unmixed metals presumed to enter into its composition; the interchanged differences represent those proportions of the unmixed metals which, combined to the extent of 10℔, are capable of displacing water equal to the mean quantity; and displacing the same quantity of water with the crown, are therefore the actual proportions of the metals contained in it; consequently, if separately multiplied into its weight, the product will be the actual quantity of each metal contained in the adulterate crown. The proof is obtained, first, by multiplying the numerator of each ratio into the quantity of water displaced by its respective metal of 10℔ weight, and dividing the sum of the products by their common denominator; a quotient, according with the mean, proves the correctness of the proportions found, from the water displaced. The same ratios are then multiplied into the weight of the crown, and their correctness proved by the sum of the parts produced being equal to the totality, or weight of the *crown, specified.*

The author cannot understand Dr. Hutton's concise notice of this *process*; he has therefore himself endeavoured to explain it.

6. Teas at 12*s.*, 10*s.*, and 6*s.*, are to be mingled with 20lb at 4*s.* per lb; what proportion of the unlimited kinds must be taken, to rate the mixture at 8*s.* per lb?

rate	12*s.*	mean,	differ.	4
	10			2
of		8*s.*	alter-	
	6			2
ingred.	4		nated	4

Ratios.		Answer.		
of limited	$\frac{4}{4}$ of 20lb=20lb	at 12*s.*	=240*s.*	
to	$\frac{2}{4}$ of 20lb=10	at 10	=100	
unlimited.	$\frac{2}{4}$ of 20lb=10	at 6	= 60	
	limited ingred., 20	at 4	= 80	
	6,0	:	48,0=8*s.* proof.	

Independent proof of alternation is unnecessary in this case, the final proof affording the same test of the whole work.

7. How much gold of 15 cáracts fine must be melted with 3*oz.* of 21*ca.*, and 5*oz.* of 22*ca.*, to make a metal of 20*ca.* fine?

rate	21*ca.*	mean,	differ. 5		22*ca.*	mean,	differ. 5
of		20*ca.*	alter-			20*ca.*	alter-
ingred.	15*ca.*		nated 1		15*ca.*		nated. 2

Ratios.

of limited	$\frac{1}{5}$ of 3*oz.*= $\frac{3}{5}$*oz.*	of 15*ca.*	= 9*ca.*
	limited ingred. 3*oz.*	of 21	= 63
to unlimited.	$\frac{2}{5}$ of 5*oz.*=2*oz.*	of 15	= 30
	limited ingred. 5*oz.*	of 22	=110
	$10\frac{3}{5}$		212
	5		5
	53	:	1060=20*ca.* proof.

Ans. $2\frac{3}{5}$*oz.* of 15*ca.*

Examples to be wrought, proved, and recited.

1. Mingled, 10lb of tea at \$·87, 6lb at \$·94, 16lb at \$·80; what is it worth per lb?

2. Mingled, 63 gallons of wine at \$1, 70*ga.* at \$1·12½, 55*ga.* at \$·97, and 54*ga.* at \$1·09; what is the price per gallon?

3. Gold bullion of 21 cáracts, and 5lb 7*oz.* in weight, is *melted with* 6lb of 17 cáracts; what is the degree of fineness?

4. Mingled 5*cw.* of sugar at $9·50 per cwt.; 7*cw.* at $10·75, 3*cw.* at $11; what is the rate per ℔?

5. Silver bullion of 8*oz.* fine, and 8℔ in weight, is melted with 6℔ of 7*oz.* fine, and 9℔ of 8½*oz.* fine; what is the fineness of the mass?

6. Silver bullion of $1 per ounce in value, and 9⅛℔ in weight, is melted with 10℔ at $1·12½, and 6¼℔ at $1·17 per ounce; what is the value of the mass per ounce?

7. How much wine at $1·50 and $·87 per gallon must be mingled, to be worth $1 per gallon?

8. Of bullion of 15, 18, 20, and 22, cáracts fine, how much must be taken, to make a mass of 19*ca.* fine?

9. Of wheats at $1·60 and $1·37½ per bushel, and of rye at $·75 per bushel, what must be the proportions intermingled, to be worth $1·50 per bushel?

10. Teas at $·76, $·93, and $1·13, are to be mingled together, and sold at $1 the pound; in what proportions?

11. Currants at $·5, $·6, $·8, and $·9, the ℔; what quantity of each must be taken, to make up a cwt. English, for sale at $·7 the ℔?

12. How much gold of 17, 19, 21, and 22, cáracts fine, must be melted together, to form a mass of ℱ 2℔ at 20*ca.* fineness?

13. How much sugar at $·7, $·9, and $·12, the ℔, must be commingled, to make a cwt. English, at $·10 the ℔?

14. Teas at $·81, $·96, $1·50, and $1·06¼, are to be mingled with 25℔ at $1·12½; what must be the quantities of the unlimited kinds, to make the mixture worth $1 the ℔?

15. How much gold of 15, 18, and 22, cáracts, must be melted with 7*oz.* of 19*ca.*, to make a mass of 20*ca.* fineness?

16. Wines at $1, $1·75, and $1·80, the gallon, are to be mingled with 20*ga.* at $2, for sale at $1·60; what must be the quantities of the unlimited kinds?

17. How much gold of 17*ca.* fineness must be melted with 5½*oz.* of 20*ca.*, and 2½*oz.* of 21*ca.*, to make a mass of 18 cáracts?

The subjects of progression, permutation, and chance, have little or no bearing on accounts; position is essentially an algebraic process, and the first mentioned subjects are best studied algebraically. Should students or instructers desire a miscellaneous collection of examples, they cannot do better than supply themselves with Mr. Hall's Arithmetical Manual, which is unrivalled in the number of exercises with solutions; and solutions there ought to be, where examples are not ranged under specific rules; yet let not the student depend on them, but chiefly on his own proofs: he may consult a guide, but he never can go securely till he can go alone.

THE END.